Entire and Meromorphic Functions

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1 An overview

A function holomorphic in the complex plane $\mathbb{C}$ is called *entire*. Such a function f has a power series expansion

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

which converges for all $z \in \mathbb{C}$.

Examples are

$$f(z) = e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n,$$

$$f(z) = \sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right),$$

or

$$f(z) = \cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right).$$

A function which is holomorphic in a domain D except for poles is called *meromorphic in D*. (The precise definition and more details will be given later.) In particular if $D = \mathbb{C}$ we just say the function is meromorphic. Functions meromorphic in $\mathbb{C}$ are precisely those which can be written as a quotient of two entire functions. (A proof of this will be given later.)

An example is

$$\tan z = \frac{\sin z}{\cos z}.$$

The following result can be seen as the starting point of the mathematical area we will discuss in this lecture.

Picard's theorem (1879). *Let f be an entire function and let $a_1, a_2 \in \mathbb{C}$ be distinct. Suppose that $f(z) \neq a_j$ for all $z \in \mathbb{C}$ and all $j \in \{1,2\}$. Then f is constant.*

Example. Let $f(z) = e^z$. Then $f(z) \neq 0$ for all $z \in \mathbb{C}$.

The example shows that we need two values a_1 and a_2 in Picard's theorem.

Let $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere. A meromorphic function f is thus a function $f\colon \mathbb{C} \to \widehat{\mathbb{C}}$, with $f(z_0) = \infty$ meaning that z_0 is a pole. An entire function f can also be considered as a meromorphic function $f\colon \mathbb{C} \to \widehat{\mathbb{C}}$ with $f(z) \neq \infty$ for all $z \in \mathbb{C}$. This leads to the following result.

Picard's theorem for meromorphic functions. *Let $f\colon \mathbb{C} \to \widehat{\mathbb{C}}$ be meromorphic and $a_1, a_2, a_3 \in \widehat{\mathbb{C}}$ distinct. Suppose that $f(z) \neq a_j$ for all $z \in \mathbb{C}$ and all $j \in \{1,2,3\}$. Then f is constant.*

The proof is by reduction to the case of entire functions; that is, the case $a_3 = \infty$. In fact if $a_j \neq \infty$ for all j, then $g = 1/(f - a_3)$ is entire and

$$g(z) = \frac{1}{f(z) - a_3} \neq b_j := \frac{1}{a_j - a_3}$$

for $j \in \{1,2\}$, so that g and hence f is constant.

Example. The tangent function satisfies $\tan z \neq \pm i$ for all $z \in \mathbb{C}$.

Around 1900, Borel and others found quantitative versions of Picard's theorem for entire functions. Let $n(r,a)$ be the number of solutions of $f(z) = a$ in the disk $\{z \in \mathbb{C}\colon |z| \leq r\}$ and let

$$M(r,f) := \max_{|z|=r} |f(z)|$$

be the *maximum modulus* of f. By Liouville's theorem, $M(r,f) \to \infty$ as $r \to \infty$ if f is non-constant.

Borel showed that $n(r,a)$ and $\log M(r,f)$ are of the same order of magnitude for all $a \in \mathbb{C}$, with at most one exception.

Borel's theorem.

$$\limsup_{r\to\infty} \frac{\log n(r,a)}{\log r} = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r}$$

for all $a \in \mathbb{C}$, with at most one exception.

Example. Consider $f(z) = e^z$. Then $M(r,f) = e^r$ so that $\log M(r,f) = r$. Let $a \in \mathbb{C}\setminus\{0\}$. Writing $a = |a|e^{i\varphi}$ we see that the solutions of $e^z = a$ are given by $z = z_k = \log|a| + i(\varphi + 2\pi k)$, where $k \in \mathbb{Z}$. It follows that

$$n(r,a) = \frac{\sqrt{r^2 - (\log|a|)^2}}{\pi} + \mathcal{O}(1)$$

and hence

$$n(r,a) \sim \frac{r}{\pi}$$

as $r \to \infty$. In particular,

$$\limsup_{r\to\infty} \frac{\log n(r,a)}{\log r} = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r} = 1$$

for all $a \in \mathbb{C}\setminus\{0\}$.

In 1924, Nevanlinna developed a theory giving quantitative estimates also for meromorphic functions, replacing $\log M(r,f)$ by a "characteristic function" $T(r,f)$. This gives stronger results also for entire functions. The theory yields far-reaching generalizations of the theorems of Picard and Borel.

The lecture will give an introduction to Nevanlinna theory, together with some further results about entire and meromorphic functions (e.g. the Weierstraß and Hadamard factorization theorems).

We will also give some applications of the theory, e.g. to differential equations or iteration. (No knowledge of these subjects is required.)

The lecture will assume familiarity with the basic results of complex function theory (as given in the course "Analysis IV"). Some results of complex function theory will be recalled.

2 Review of complex function theory

Let $D \subset \mathbb{C}$ be a domain (i.e. open and connected) and let $f\colon D \to \mathbb{C}$ be a function. A substantial part of a basic complex function course consists of showing that the following four properties are equivalent:

(i) f is complex differentiable at every point of D; that is, for all $z_0 \in D$ the limit

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists;

(ii) with $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ the function given by

$$(x, y) \mapsto (u(x + iy), v(x + iy))$$

is differentiable in the real sense and satisfies the *Cauchy-Riemann differential equations*

$$u_x = v_y, \qquad u_y = -v_y;$$

(iii) f is continuous and for every null-homologous, closed, piecewise smooth curve γ in D we have

$$\int_\gamma f(z)dz = 0,$$

here γ is called *null-homologous* if the *winding number*

$$n(\gamma, z) = \frac{1}{2\pi i} \int_\gamma \frac{d\zeta}{\zeta - z}$$

satisfies $n(\gamma, z) = 0$ for all $z \in \mathbb{C} \setminus D$;

(iv) f can be developed into a power series around any point in D; that is, for all $z_0 \in D$ there exists $r > 0$ and a sequence $(c_n)_{n \geq 0}$ in $\mathbb{C}$ such that

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

for $|z - z_0| < r$.

A function having one (and hence all) of these properties is called *holomorphic*.

In (iv) we can take any $r > 0$ such that

$$D(z_0, r) := \{z \in \mathbb{C}\colon |z - z_0| < r\} \subset D.$$

The radius of convergence of the power series may be greater than r. The coefficients have the representations

$$c_n = \frac{1}{n!} f^{(n)}(z_0) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = \rho} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$$

if $0 < \rho < r$.

In (iii), instead of considering all integrals over arbitrary nullhomologous curves, it suffices to consider integrals over the boundaries of triangles in D; that is, if $\int_{\partial\Delta} f(z)dz = 0$ for every closed triangle Δ contained in D, then f is holomorphic. This result is known as Morera's Theorem.

If $K \subset \widehat{\mathbb{C}}$, not necessarily open, and if $f\colon K \to \mathbb{C}$, then we say that f is meromorphic (or holomorphic) in K if there exists a domain $D \supset K$ such that f extends to a meromorphic (or holomorphic) function $f\colon D \to K$.

For $K \subset \mathbb{C}$ we denote by $\overline{K}$ the closure of K. For $a \in \mathbb{C}$ and $r > 0$ we put

$$\overline{D}(a, r) = \overline{D(a, r)} = \{z \in \mathbb{C}\colon |z - a| \leq r\}.$$

Cauchy's integral formula says that if $f\colon D \to \mathbb{C}$ is holomorphic and γ a nullhomologous, closed, piecewise smooth curve in D, then

$$n(\gamma, z)f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta.$$

In particular, if f is holomorphic in $\overline{D}(z_0, r)$, then

$$f(z) = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(\zeta)}{\zeta - z} d\zeta$$

for $z \in D(z_0, r)$. Differentiating this we obtain

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_{|z-z_0|=r} \frac{f(\zeta)}{(\zeta - z)^{k+1}} d\zeta$$

If D is a domain, $z_0 \in D$ and $f\colon D \setminus \{z_0\} \to \mathbb{C}$ is holomorphic, then z_0 is called an *isolated singularity* of f. It is called

- *removable*, if $\lim_{z\to z_0} f(z) \in \mathbb{C}$. By putting $f(z_0) = \lim_{z\to z_0} f(z)$ the function f extends to a holomorphic function $f\colon D \to \mathbb{C}$;
- a *pole*, if $\lim_{z\to z_0} |f(z_0)| = \infty$;
- *essential* otherwise.

In the case of a pole we write $f(z_0) = \infty$ and with $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ thus consider f as a function $f\colon D \to \widehat{\mathbb{C}}$.

Given a function f, we usually assume that removable singularities have already been "removed" and that at poles the value ∞ has been assigned.

Example. Consider

$$f\colon \mathbb{C} \to \widehat{\mathbb{C}},\ f(z) = \frac{\sin(\pi z)}{\sin(\pi z^2)}.$$

The expression defining f has isolated singularities at the points $\pm\sqrt{k}$ and $\pm i\sqrt{k}$, for all $k \in \mathbb{Z}$. We have

$$f\left(\sqrt{2}\right) = \infty,$$

$$f(0) = \lim_{z\to z} \frac{\sin(\pi z)}{\sin(\pi z^2)} = \lim_{z\to 0} \frac{\pi\cos(\pi z)}{2\pi z\cos(\pi z^2)} = \infty$$

and

$$f(1) = \lim_{z\to 1} \frac{\sin(\pi z)}{\sin(\pi z^2)} = \frac{1}{2}.$$

Rational functions (i.e., quotients of polynomials) are meromorphic in $\mathbb{C}$ (even in $\widehat{\mathbb{C}}$). Rational functions of degree 1 are called *Möbius transformations* (or fractional linear transformations). They are bijective maps from $\widehat{\mathbb{C}}$ to $\widehat{\mathbb{C}}$.

3 Jensen's formula

Theorem 3.1. *Let $r > 0$ and let f be holomorphic in $\overline{D}(0,r)$. Then*

$$f(0) = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})d\theta,$$

$$\operatorname{Re} f(0) = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re} f(re^{i\theta})d\theta,$$

and

$$\operatorname{Im} f(0) = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Im} f(re^{i\theta})d\theta.$$

Proof. Let

$$\gamma\colon [0,2\pi] \to \mathbb{C},\ \gamma(\theta) = re^{i\theta},$$

be the standard parametrization of $\partial D(0,r)$. Cauchy's integral formula yields

$$\begin{aligned} f(0) &= \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z}dz \\ &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(\gamma(\theta))}{\gamma(\theta)}\gamma'(\theta)d\theta \\ &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(re^{i\theta})}{re^{i\theta}} ire^{i\theta}d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})d\theta. \end{aligned}$$

This is the first claim. The second and third claims follow by taking real and imaginary parts. □

Remark. The result is also known as the *mean value property.*

Theorem 3.2 (Jensen's formula)**.** *Let $r > 0$ and f meromorphic in $\overline{D}(0,r)$. Let $a_1,\dots,a_m$ be the zeros of f and $b_1,\dots,b_n$ the poles of f in $D(0,r)\backslash\{0\}$, counted with multiplicity. Let*

$$f(z) = \sum_{k=\ell}^{\infty} c_k z^k$$

be the Laurent series expansion of f near 0, with $c_\ell \neq 0$. Then

$$\log|c_\ell| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|d\theta - \sum_{j=1}^{m}\log\frac{r}{|a_j|} + \sum_{k=1}^{n}\log\frac{r}{|b_k|} - \ell\log r.$$

Remark. If $f(0) \neq 0, \infty$, then we have $\ell = 0$ and $c_\ell = \log|f(0)|$ in Theorem 3.2.

We first state the following lemma.

Lemma 3.1. *Let $r > 0$, $a \in D(0,r)$ and*

$$\varphi\colon\ \mathbb{C} \to \widehat{\mathbb{C}}, \qquad \varphi(z) = \frac{r(z-a)}{r^2 - \overline{a}z}.$$

Then φ is holomorphic in $\overline{D}(0,r)$ and $|\varphi(z)| = 1$ for $z \in \partial D(0,r)$.

The proof is left as an exercise.

Proof of Theorem 3.2. We will start with a special case and then increase the generality step by step.

(1) Suppose first that f has neither zeros or poles in $\overline{D}(0,r)$; that is, f is holomorphic in $\overline{D}(0,r)$ and $f(z) \neq 0$ for $z \in \overline{D}(0,r)$. Then there exists a function g holomorphic in $\overline{D}(0,r)$ such that $f = \exp\circ g$. (The existence of such a function g is standard, but we repeat the argument: First choose $R > r$ such that $f\colon D(0,R) \to \mathbb{C}$ is holomorphic and without zeros. Then there exists $G\colon D(0,R) \to \mathbb{C}$ with $G' = f'/f$. For example, one may put $G(z) = \int_0^z \frac{f'(\zeta)}{f(\zeta)}d\zeta$. We then choose $c \in \mathbb{C}$ such that $g(z) = G(z) + c$ satisfies $e^{g(0)} = f(0)$. The function h defined by $h(z) = f(z)/e^{g(z)} = f(z)e^{-g(z)}$ then satisfies $h(0) = 1$ and $h' = f'e^{-g} - fg'e^{-g} = 0$ and thus $h(z) = 1$ for all $z \in D(0,R)$. Thus $f = e^g$.)

   It follows that $|f| = e^{\operatorname{Re} g}$ and thus $\log|f| = \operatorname{Re} g$. The conclusion now follows from Theorem 3.1.

(2) We allow zeros and poles in $D(0,r)\backslash\{0\}$, but assume that f has no other zeros and poles in $\overline{D}(0,r)$; that is, $f(0) \neq 0, \infty$ and $f(z) \neq 0, \infty$ for $z \in \partial D(0,r)$.

   Let $\varphi = \varphi_a$ be as in Lemma 3.1, that is,

$$\varphi_a(z) = \frac{r(z-a)}{r^2 - \overline{a}z},$$

   and consider the function h given by

$$h(z) = f(z)\frac{\prod_{k=1}^{n}\varphi_{b_k}(z)}{\prod_{j=1}^{m}\varphi_{a_j}(z)}.$$

Then h is holomorphic in $\overline{D}(0,r)$ and has no zeros in $\overline{D}(0,r)$. Moreover, we have $|h(z)| = |f(z)|$ for $z \in \partial D(0,r)$ by Lemma 3.1. It follows, using the result of part (1) of the proof, that

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|d\theta &= \frac{1}{2\pi}\int_0^{2\pi} \log|h(re^{i\theta})|d\theta \\ &= \log|h(0)| \\ &= \log|f(0)| + \sum_{k=1}^{n} \log|\varphi_{b_k}(0)| - \sum_{j=1}^{m} \log|\varphi_{a_j}(0)| \\ &= \log|f(0)| + \sum_{k=1}^{n} \log\frac{|b_k|}{r} - \sum_{j=1}^{m} \log\frac{|a_j|}{r},\end{aligned}$$

from which the conclusion follows.

(3) Now we allow that $f(0) = 0$ or $f(z) = \infty$, but still assume that $f(z) \neq 0, \infty$ for $z \in \partial D(0,r)$. With ℓ and c_ℓ as in the statement of the theorem we consider

$$h(z) = \frac{f(z)}{z^\ell}.$$

Thus $h(0) = c_\ell \neq 0, \infty$. Since $f(z) = h(z)z^\ell$, part (2) of the proof yields that

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|d\theta &= \frac{1}{2\pi}\int_0^{2\pi} \log|h(re^{i\theta})|d\theta + \ell\log r \\ &= \log|h(0)| - \sum_{k=1}^{n} \log\frac{r}{|b_k|} + \sum_{j=1}^{m} \log\frac{r}{|a_j|} + \ell\log r, \\ &= \log|c_\ell| - \sum_{k=1}^{n} \log\frac{r}{|b_k|} + \sum_{j=1}^{m} \log\frac{r}{|a_j|} + \ell\log r\end{aligned}$$

as claimed.

(4) We finally consider the general case, i.e., zeros and poles are also allowed on $\partial D(0,r)$. Then

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|d\theta$$

is an improper integral. The existence of this integral is easy to show.

To prove the formula claimed, there are two possibilities:

- reduce to part (3) by multiplying f with factors $z - b_k$ or dividing by $z - a_j$, similarly as in part (2);
- replace r by $r - \varepsilon$ and take the limit as $\varepsilon \to 0$.

We omit the details. □

We want to rewrite the sums occurring in Jensen's formula.

Lemma 3.2. *Let $r > 0$ and $c_1, \dots, c_m \in \overline{D}(0,r) \setminus \{0\}$. For $0 \le t \le r$ let $n(t)$ denote the number of c_j in $\overline{D}(0,t)$. Then*

$$\sum_{j=1}^{m} \log \frac{r}{|c_j|} = \int_0^r \frac{n(t)}{t} dt.$$

Proof. Without loss of generality we may assume that $0 < |c_1| \le |c_2| \le \dots \le |c_m| \le r$. Using

$$\log \frac{r}{|c_j|} = \log r - \log |c_j| = \int_{|c_j|}^{r} \frac{dt}{t}$$

we find that

$$\begin{aligned}
\sum_{j=1}^{m} \log \frac{r}{|c_j|} = \sum_{j=1}^{m} \int_{|c_j|}^{r} \frac{dt}{t} &= \sum_{j=1}^{m} \int_{|c_j|}^{|c_m|} \frac{dt}{t} + m \int_{|c_m|}^{r} \frac{dt}{t} \\
&= \sum_{j=1}^{m} \sum_{k=j}^{m-1} \int_{|c_k|}^{|c_{k+1}|} \frac{dt}{t} + m \int_{|c_m|}^{r} \frac{dt}{t} \\
&= \sum_{k=1}^{m-1} \sum_{j=1}^{k} \int_{|c_k|}^{|c_{k+1}|} \frac{dt}{t} + m \int_{|c_m|}^{r} \frac{dt}{t} \\
&= \sum_{k=1}^{m-1} k \int_{|c_k|}^{|c_{k+1}|} \frac{dt}{t} + m \int_{|c_m|}^{r} \frac{dt}{t} \\
&= \sum_{k=1}^{m-1} \int_{|c_k|}^{|c_{k+1}|} \frac{n(t)}{t} dt + \int_{|c_m|}^{r} \frac{dt}{t} \\
&= \int_{|c_1|}^{r} \frac{n(t)}{t} dt = \int_0^r \frac{n(t)}{t} dt.
\end{aligned}$$

□

Remark. The above considerations may be elegantly described using the Riemann-Stieltjes integral. We discuss these integrals in a short excursion:

For a function $f\colon [a,b] \to \mathbb{C}$, a partition of the interval $[a,b]$ given by $a = x_0 < x_1 < \dots < x_n = b$ and points $\xi_j \in [x_{j-1}, x_j]$, the corresponding Riemann sum is given by

$$S(f, (x_j), (\xi_j)) = \sum_{j=1}^{n} f(\xi_j)(x_j - x_{j-1}).$$

The Riemann integral is obtained by a limit process:

$$S(f, (x_j), (\xi_j)) \to \int_a^b f(x)dx \quad \text{as } \max_j |x_j - x_{j-1}| \to 0.$$

Let, in addition, a function $g\colon [a,b] \to \mathbb{C}$ be given. The *Riemann-Stieltjes sum* is defined by

$$S(f, g, (x_j), (\xi_j)) = \sum_{j=1}^{n} f(\xi_j) \left(g(x_j) - g(x_{j-1}) \right).$$

If these sums tend to a limit as $\max_j |x_j - x_{j-1}| \to 0$, the limit is called Riemann-Stieltjes integral and denoted by

$$\int_a^b f(x)dg(x).$$

It exists for example if f is continuous and g is of *bounded variation*, i.e.,

$$L(g) := \sup_{(x_j)} \sum_j |g(x_j) - g(x_{j-1})| < \infty.$$

This holds in particular if g is monotone.

If $g \in C^1[a,b]$ and $f \in C[a,b]$, then

$$\int_a^b f(x)dg(x) = \int_a^b f(x)g'(x)dx.$$

But Riemann-Stieltjes integrals are of interest also if g is discontinuous. For example, if $f\colon [a,b] \to \mathbb{C}$ is continuous, $x_0 \in (a,b)$, $g(x) = \alpha$ for $x < x_0$ and $g(x) = \beta$ for $x \geq x_0$, then

$$\int_a^b f(x)dg(x) = f(x_0)(\beta - \alpha).$$

The rule of integration by parts takes the following form:

$$\int_a^b f(x)dg(x) = f(x)g(x)\Big|_{x=a}^{x=b} - \int_a^b g(x)df(x).$$

For $g(t) = n(t)$ and $f(t) = \log(r/t) = \log r - \log t$ this is precisely the formula from Lemma 3.2.

Integrals of the form $\int_0^r \frac{a(t)}{t}dt$ will occur repeatedly. It may be helpful to write them (or at least think of them) as integrals of the form $\int_0^r a(t)d\log t$.

Definition 3.1. Let $r > 0$ and f meromorphic in $\overline{D}(0,r)$. For $0 \leq t \leq r$ let $n(t,f)$ denote the number of poles of f in $\overline{D}(0,t)$, counted according to multiplicity. Then

$$N(r,f) = \int_0^r \frac{n(t,f) - n(0,f)}{t}dt + n(0,f)\log r$$

is called the (Nevanlinna) counting function of the poles of f.

Remark. (1) The zeros of f are poles of $1/f$. For $a \in \mathbb{C}$ the zeros of $f - a$ are called a-points of f. They are the poles of $1/(f-a)$. Thus $n(r, 1/(f-a))$ and $N(r, 1/(f-a))$ count the a-points of f.

Sometimes, if the function f considered is clear from the context, we write $n(r,a)$ and $N(r,a)$ instead of $n(r,1/(f-a))$ and $N(r,1/(f-a))$. Then we also write $n(r,\infty)$ and $N(r,\infty)$ instead of $n(r,f)$ and $N(r,f)$.

(2) If $f(0) \neq \infty$, then $n(0,f) = 0$ and

$$N(r,f) = \int_0^r \frac{n(t,f)}{t}dt.$$

For $0 < r_0 < r$ we always have

$$\begin{aligned} N(r,f) &= \int_0^{r_0} \frac{n(t,f)-n(0,f)}{t}dt + \int_{r_0}^r \frac{n(t,f)-n(0,f)}{t}dt + n(0,\infty)\log r \\ &= N(r_0,f) + \int_{r_0}^r \frac{n(t)}{t}dt. \end{aligned}$$

Using Definition 3.1 Jensen's formula takes the following form.

Theorem 3.3. *Let $r > 0$ and f meromorphic in $\overline{D}(0,r)$. Let c_ℓ be the first non-zero coefficient in the Laurent series of f as in Jensen's formula (Theorem 3.2). Then*

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|d\theta = N\left(r, \frac{1}{f}\right) - N(r,f) + \log|c_\ell|.$$

Proof. If $f(0) \neq 0, \infty$ so that $c_\ell = f(0)$ (and $\ell = 0$) this follows directly from Theorem 3.2 and Lemma 3.2.

Suppose that $f(0) = \infty$. Then $\ell < 0$ and 0 is a pole of f of order $-\ell$. Thus $n(0,f) = -\ell$. Moreover, if $b_1, \ldots, b_n$ are the poles of f in $\overline{D}(0,r) \setminus \{0\}$, then

$$\sum_{k=1}^n \log\frac{r}{|b_k|} = \int_0^r \frac{n(t,f)-n(0,f)}{t}dt$$

by Lemma 3.2. Thus

$$\begin{aligned} N(r,f) &= \int_0^r \frac{n(t,f)-n(0,f)}{t}dt + n(0,f)\log r \\ &= \sum_{k=1}^n \log\frac{r}{|b_k|} - \ell\log r \end{aligned}$$

and the conclusion follows with Theorem 3.2. The case that $f(0) = 0$ is analogous. Here $n(0,1/f) = \ell$. □

Definition 3.2. Let f be entire and $r > 0$. Then

$$M(r,f) := \max_{|z|=r}|f(z)|$$

is called the *maximum modulus* of f.

Remark. The maximum principle yields that $M(r,f)$ strictly increases with r if f is non-constant.

Theorem 3.4. *Let f be entire, $r > 0$ and $f(0) \neq 0$. Then*

$$N\left(r, \frac{1}{f}\right) \leq \log M(r, f) - \log |f(0)|.$$

Proof. This follows directly from Theorem 3.3, since $N(r, f) = 0$ and $c_\ell = f(0)$. □

Theorems like Theorem 3.3 or Theorem 3.4 are the reason that we passed from the "natural" function $n(r, f)$ counting the poles to the function $N(r, f)$. It is, however, also possible to obtain estimates for $n(r, f)$ from this.

Lemma 3.3. *Let $1 \leq r \leq R$ and f meromorphic in $\overline{D}(0, R)$. Then*

$$N(r, f) \leq n(r, f) \log r + N(1, f)$$

and

$$N(R, f) \geq \left(\log \frac{R}{r}\right) n(r, f).$$

Proof. The first claim follows with

$$\begin{aligned} N(r, f) &= \int_1^r \frac{n(t, f)}{t} dt + N(1, f) \\ &\leq n(r, f) \int_1^r \frac{dt}{t} + N(1, f) \\ &= n(r, f) \log r + N(1, f), \end{aligned}$$

the second one with

$$\begin{aligned} N(R, f) &= \int_1^R \frac{n(t, f)}{t} dt + N(1, f) \\ &\geq \int_1^R \frac{n(t, f)}{t} dt \\ &\geq \int_r^R \frac{n(t, f)}{t} dt \\ &\geq n(r, f) \int_r^R \frac{dt}{t} \\ &= n(r, f) \log \frac{R}{r}. \end{aligned}$$

□

Theorem 3.5. *Let f be entire with $f(0) \neq 0$. Let $r > 0$ and $K > 1$. Then*

$$n\left(r, \frac{1}{f}\right) \leq \frac{1}{\log K} \left(\log M(Kr, f) - \log |f(0)|\right).$$

Proof. With $R = Kr$ in Lemma 3.3 we deduce from Theorem 3.4 that

$$\begin{aligned} n\left(r, \frac{1}{f}\right) &\leq \frac{1}{\log K} N\left(Kr, \frac{1}{f}\right) \\ &\leq \frac{1}{\log K} \left(\log M(Kr, f) - \log |f(0)|\right). \end{aligned}$$

□

Remark. Suppose that $\log M(r, f) = \mathcal{O}(r^\rho)$ as $r \to \infty$ for some $\rho > 0$, say

$$\log M(r, f) \le \tau r^\rho \quad \text{for} \quad r \ge r_0.$$

Then

$$n\left(r, \frac{1}{f}\right) \le \frac{1}{\log K} N\left(Kr, \frac{1}{f}\right) + \mathcal{O}(1) \le \frac{\tau K^\rho}{\log K} r^\rho + \mathcal{O}(1).$$

First this follows only if $f(0) \ne 0$, but considering $h(z) = f(z)/z^p$ instead of f if 0 is a zero of f of multiplicity p and noting that $n(r, 1/f) = n(r, h) + p$ this also holds if $f(0) = 0$.

It follows that

$$n\left(r, \frac{1}{f}\right) = \mathcal{O}(r^\rho) \quad \text{as } r \to \infty.$$

With $K = e^{1/\rho}$ we actually have

$$n\left(r, \frac{1}{f}\right) \le e\rho\tau r^\rho + \mathcal{O}(1)$$

as $r \to \infty$.

More generally,

$$n\left(r, \frac{1}{f - a}\right) = \mathcal{O}(r^\rho) \quad \text{as } r \to \infty \text{ for all } a \in \mathbb{C}.$$

So in some sense $n(r, 1/(f - a))$ does not grow faster than $\log M(r, f)$. We will see that for all $a \in \mathbb{C}$, with at most one exception, $n(r, 1/(f - a))$ will grow of the same order of magnitude as $\log M(r, f)$.

4 The Nevanlinna characteristic and the first fundamental theorem

We define $\log^+ \colon \mathbb{R} \to \mathbb{R}$ by

$$\log^+ x = \begin{cases} \log x, & x \ge 1, \\ 0, & x \le 1. \end{cases}$$

Definition 4.1. Let $r > 0$ and f meromorphic in $\overline{D}(0, r)$. Then

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta$$

is called the *proximity function* and

$$T(r, f) = N(r, f) + m(r, f)$$

is called the *(Nevanlinna) characteristic* of f.

Distinguishing the cases $x \geq 1$ and $x < 1$ we see that

$$\log x = \log^+ x - \log^+ \frac{1}{x}.$$

This yields that

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta = m(r, f) - m\left(r, \frac{1}{f}\right).$$

Inserting this into the formula of Theorem 3.3 yields

$$m(r, f) - m\left(r, \frac{1}{f}\right) = N\left(r, \frac{1}{f}\right) - N(r, f) + \log |c_\ell|$$

and thus

$$T(r, f) = T\left(r, \frac{1}{f}\right) + \log |c_\ell|.$$

We will also consider $T(r, 1/(f - a))$ for $a \in \mathbb{C}$. Before doing so, we collect some properties of the function $\log^+$.

Lemma 4.1. *Let $x_1, \ldots, x_n \geq 0$ and $x \geq 0$. Then*

(*i*) $|\log x| = \log^+ x + \log^+ \frac{1}{x}$;

(*ii*) $\log^+ \left(\prod_{j=1}^n x_j \right) \leq \sum_{j=1}^n \log^+ x_j$;

(*iii*) $\log^+ \left(\sum_{j=1}^n x_j \right) \leq \sum_{j=1}^n \log^+ x_j + \log n$;

(*iv*) $\left| \log^+ x_1 - \log^+ x_2 \right| \leq \log^+ |x_1 - x_2| + \log 2$.

Proof. Conclusion (*i*) follows by distinguishing the cases $x \geq 1$ and $x < 1$.

Conclusion (*ii*) is clear if $\prod_{j=1}^n x_j \leq 1$ and follows otherwise since then

$$\log^+ \left(\prod_{j=1}^n x_j \right) = \log \left(\prod_{j=1}^n x_j \right) = \sum_{j=1}^n \log x_j \leq \sum_{j=1}^n \log^+ x_j.$$

Conclusion (*iii*) follows since

$$\begin{aligned} \log^+ \left(\sum_{j=1}^n x_j \right) &\leq \log^+ (n \cdot \max_j x_j) \\ &\leq \log^+ n + \log^+ (\max_j x_j) \\ &\leq \log n + \log^+ \left(\sum_{j=1}^n x_j \right). \end{aligned}$$

To prove (*iv*), we assume without loss of generality that $x_1 \geq x_2$. Then, by (*iii*),

$$\begin{aligned} \left|\log^+ x_1 - \log^+ x_2\right| &= \log^+ x_1 - \log^+ x_2 \\ &= \log^+(x_1 - x_2 + x_2) - \log^+ x_2 \\ &\leq \log^+(x_1 - x_2) + \log^+ x_2 + \log 2 - \log^+ x_2 \\ &= \log^+ |x_1 - x_2| + \log 2. \end{aligned}$$

□

Theorem 4.1 (First Fundamental Theorem)**.** *Let $r > 0$, f meromorphic in $\overline{D}(0,r)$ and $a \in \mathbb{C}$. Let $c(a)$ be the first non-zero coefficient in the Laurent series of $f - a$ around 0. Then*

$$T\left(r, \frac{1}{f-a}\right) = T(r,f) - \log|c(a)| + \varphi(r,a)$$

where $|\varphi(r,a)| \leq \log^+|a| + \log 2$.

Proof. For $a = 0$ this is the formula stated before Lemma 4.1.

For $a \neq 0$ this formula yields that

$$T(r, f-a) = T\left(r, \frac{1}{f-a}\right) + \log|c(a)|.$$

Hence

$$\begin{aligned} T\left(r, \frac{1}{f-a}\right) &= T(r, f-a) - \log|c(a)| \\ &= N(r, f-a) + m(r, f-a) - \log|c(a)| \\ &= N(r,f) + m(r,f) + \varphi(r,a) - \log|c(a)| \\ &= T(r,f) + \varphi(r,a) - \log|c(a)| \end{aligned}$$

with

$$\varphi(r,a) = m(r, f-a) - m(r,f).$$

By Lemma 4.1, (*iv*), we have

$$\left|\log^+|f(re^{i\theta}) - a| - \log^+|f(re^{i\theta})|\right| \leq \log^+|a| + \log 2$$

and thus

$$\varphi(r,a) \leq \log^+|a| + \log 2.$$

□

Remark. The main point is that the difference between $T(r,f)$ and $T(r, 1/(f-a))$ is bounded by a constant independent of r.

We will see soon that for f meromorphic in $\mathbb{C}$ and non-constant we have $T(r,f) \to \infty$ as $r \to \infty$. The first fundamental theorem says that

$$N\left(r, \frac{1}{f-a}\right) + m\left(r, \frac{1}{f-a}\right) = T\left(r, \frac{1}{f-a}\right) = T(r,f) + \mathcal{O}(1)$$

as $r \to \infty$.

The interpretation is as follows:

– $N\left(r, \dfrac{1}{f-a}\right)$ is large if f has many a-points;

– $m\left(r, \dfrac{1}{f-a}\right)$ is large if f is close to a on some part of $\partial D(0, r)$.

If $T(r, f)$ is large we must have one of these two possibilities.

The second fundamental theorem will say that for "most" values of a the first alternative holds; that is, the term $m(r, 1/(f-a))$ is small and thus $N(r, 1/(f-a))$ is large.

Example 4.1. Let $f(z) = e^z$. Since f is entire, $N(r, f) = 0$. Moreover,

$$\begin{aligned} m(r, f) &= \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| e^{re^{i\theta}} \right| d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left(e^{r\cos\theta} \right) d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \max\{0, r\cos\theta\} d\theta \\ &= \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} r\cos\theta \, d\theta \\ &= \frac{r}{\pi}. \end{aligned}$$

Thus

$$T(r, f) = m(r, f) = \frac{r}{\pi}.$$

Analogously,

$$N\left(r, \frac{1}{f}\right) = 0 \quad \text{and} \quad m\left(r, \frac{1}{f}\right) = \frac{r}{\pi}.$$

We saw in the introduction that if $a \in \mathbb{C} \setminus \{0\}$, then

$$\left| n\left(r, \frac{1}{f-a}\right) - \frac{\sqrt{r^2 - (\log|a|)^2}}{\pi} \right| \leq K$$

for some constant K if $r \geq |\log|a||$.

Since

$$0 \leq r - \sqrt{r^2 - c^2} = \frac{r^2 - (r^2 - c^2)}{r + \sqrt{r^2 - c^2}} = \frac{c^2}{r + \sqrt{r^2 - c^2}} \leq \frac{c^2}{r} \leq c$$

for $c \geq 0$ we have $\sqrt{r^2 - (\log|a|)^2}/\pi = r/\pi + \mathcal{O}(1)$ and thus

$$n\left(r, \frac{1}{f-a}\right) = \frac{r}{\pi} + \mathcal{O}(1).$$

It follows that

$$\begin{aligned} N\left(r, \frac{1}{f-a}\right) &= N\left(1, \frac{1}{f-a}\right) + \int_1^r n\left(t, \frac{1}{f-a}\right) \frac{dt}{t} \\ &= \frac{r}{\pi} + \mathcal{O}(\log r) \\ &= T(r,f) + \mathcal{O}(\log r). \end{aligned}$$

The first fundamental theorem implies that

$$m\left(r, \frac{1}{f-a}\right) = \mathcal{O}(\log r).$$

One can show that in fact $m(r, 1/(f-a)) = \mathcal{O}(1)$.

Example 4.2. Let f be a rational function, say $f(z) = p(z)/q(z)$ with polynomials p and q without common zeros. Let

$$p(z) = a_\ell z^\ell + \ldots + a_0,$$

$$q(z) = b_m z^m + \ldots + b_0,$$

with $a_\ell \neq 0$ and $b_m \neq 0$. By the fundamental theorem of algebra, q has m zeros. Thus f has m poles, so there exists $r_0 > 0$ with $n(r,f) = m$ for $r \geq r_0$. Hence

$$N(r,f) = N(r_0, f) + \int_{r_0}^r m \frac{dt}{t} = m \log r + C$$

with a constant C.

We also have

$$|p(z)| \sim |a_\ell| \cdot |z|^\ell \quad \text{and} \quad |q(z)| \sim |b_m| \cdot |z|^m$$

and hence

$$|f(z)| \sim \frac{|a_\ell|}{|b_m|} |z|^{\ell - m}$$

as $|z| \to \infty$.

If $\ell \leq m$, then $|f(z)| = \mathcal{O}(1)$ as $|z| \to \infty$ and hence $m(r,f) = \mathcal{O}(1)$ as $r \to \infty$. Hence $T(r,f) = N(r,f) + \mathcal{O}(1) = m \log r + \mathcal{O}(1)$.

If $\ell > m$, then, since

$$\log |f(z)| = (\ell - m) \log |z| + \log \frac{|a_\ell|}{|b_m|} + o(1)$$

as $|z| \to \infty$, we have

$$m(r,f) = (\ell - m) \log r + \mathcal{O}(1)$$

as $r \to \infty$. Hence

$$\begin{aligned} T(r,f) &= N(r,f) + m(r,f) \\ &= m \log r + (\ell - m) \log r + \mathcal{O}(1) \\ &= \ell \log r + \mathcal{O}(1). \end{aligned}$$

In both cases we thus have

$$T(r,f) = \max\{\ell, m\} \cdot \log r + \mathcal{O}(1) = \deg(f) \cdot \log r + \mathcal{O}(1).$$

Had we only aimed at the last equation, we could have reduced the case $\ell > m$ to the case $\ell \leq m$ by passing from f to $1/f$ and using the first fundamental theorem.

5 Properties of the Nevanlinna characteristic

Theorem 5.1. *Let $f_1, \dots, f_n$ be meromorphic and $r \geq 1$. Then*

(i) $T\left(r, \prod_{j=1}^{n} f_j\right) \leq \sum_{j=1}^{n} T(r, f_j),$

(ii) $T\left(r, \sum_{j=1}^{n} f_j\right) \leq \sum_{j=1}^{n} T(r, f_j) + \log n.$

Proof. The corresponding results for $m(r,\cdot)$ follow directly from Lemma 4.1.

To prove the corresponding estimates for $N(r,\cdot)$, we note that $\prod_{j=1}^{n} f_j$ can have a pole only where one of the function f_j has a pole. One obtains

$$n\left(0, \prod_{j=1}^{n} f_j\right) \leq \sum_{j=1}^{n} n(0, f_j)$$

and

$$n\left(t, \prod_{j=1}^{n} f_j\right) - n\left(0, \prod_{j=1}^{n} f_j\right) \leq \sum_{j=1}^{n} \left(n(t, f_j) - n(0, f_j)\right).$$

Integration yields

$$N\left(r, \prod_{j=1}^{n} f_j\right) \leq \sum_{j=1}^{n} N(r, f_j).$$

Analogously we obtain

$$N\left(r, \sum_{j=1}^{n} f_j\right) \leq \sum_{j=1}^{n} N(r, f_j).$$

Together with the results for $m(r,\cdot)$ this yields the conclusion. □

Theorem 5.2. *Let f be meromorphic and let M be a Möbius transformation (i.e., a rational function of degree 1). Then*

$$T(r, M \circ f) = T(r,f) + \mathcal{O}(1).$$

Proof. Every Möbius transformation can be written as a composition of Möbius transformation of the following three types:

(*i*) $z \mapsto 1/z$ (inversion)

(*ii*) $z \mapsto z + c$, $c \in \mathbb{C}$ (translation)

(*iii*) $z \mapsto c \cdot z$, $c \in \mathbb{C} \setminus \{0\}$ (dilation/rotation)

For example, if $M(z) = \dfrac{\alpha z + \beta}{\gamma z + \delta}$ with $\gamma \neq 0$ (and $\alpha\delta - \beta\gamma \neq 0$ since M is non-constant), then

$$M(z) = -\frac{\alpha\delta - \beta\gamma}{\gamma} \cdot \frac{1}{\gamma z + \delta} + \frac{\alpha}{\gamma},$$

so M is composition of $z \mapsto \gamma z$, $z \mapsto z + \delta$, $z \mapsto -(\alpha\delta - \beta\gamma)/\gamma z$ and $z \mapsto z + \alpha/\gamma$.

It then suffices to prove the conclusion for Möbius transformations of types (*i*), (*ii*) and (*iii*). For type (*i*) this follows from the first fundamental theorem, and cases (*ii*) and (*iii*) are easy. □

Theorem 5.3. *Let f be meromorphic and P a polynomial of degree $d \geq 1$. Then*

$$T(r, P \circ f) = d \cdot T(r, f) + \mathcal{O}(1).$$

Proof. We have $n(r, P \circ f) = d \cdot n(r, f)$ and thus

$$N(r, P \circ f) = d \cdot N(r, f).$$

Let $P(z) = a_d z^d + \ldots + a_0$. Then there exists $r_0 > 1$ with

$$1 < \frac{1}{2}|a_d| \cdot |z|^d \leq |P(z)| \leq \frac{3}{2}|a_d| \cdot |z|^d \quad \text{for } |z| \geq r_0.$$

Thus

$$\log \frac{|a_d|}{2} + d \cdot \log^+ |z| \leq \log^+ |P(z)| \leq d \cdot \log^+ |z| + \log^+ \frac{3 \log d}{2}$$

for $|z| \geq r_0$. This implies that there exists a constant C such that

$$\big| \log^+ |P(z)| - d \cdot \log^+ |z| \big| \leq C$$

for all $z \in \mathbb{C}$. Hence

$$|m(r, P \circ f) - d \cdot m(r, f)| \leq C$$

and thus

$$|T(r, P \circ f) - d \cdot T(r, f)| \leq C.$$

□

Theorem 5.4 (Cartan's formula)**.** *Let f be meromorphic, $f(0) \neq \infty$ and $r > 0$. Then*

$$T(r, f) = \frac{1}{2\pi} \int_0^{2\pi} N\left(r, \frac{1}{f - e^{i\varphi}}\right) d\varphi + \log^+ |f(0)|.$$

For the proof we will use the following lemma.

Lemma 5.1. *Let $a \in \mathbb{C}$. Then*

$$\log^+ |a| = \frac{1}{2\pi} \int_0^{2\pi} \log |a - e^{i\varphi}| d\varphi.$$

Proof. We apply Jensen's formula for $f(z) = a - z$ and $r = 1$. Distinguishing the cases $|a| > 1$ and $|a| \le 1$ we obtain the conclusion. □

Proof of Theorem 5.4. Theorem 3.3 yields that if $f(0) \neq e^{i\varphi}$ and $r > 0$, then

$$\frac{1}{2\pi} \int_0^{2\pi} \log \left| f(re^{i\theta}) - e^{i\varphi} \right| d\theta = N\left(r, \frac{1}{f - e^{i\varphi}}\right) - N(r, f) + \log |f(0) - e^{i\varphi}|.$$

Integrating with respect to φ we obtain, using Lemma 5.1,

$$\begin{aligned} &\frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2\pi} \int_0^{2\pi} \log \left| f(re^{i\theta}) - e^{i\varphi} \right| d\theta d\varphi \\ &= \frac{1}{2\pi} \int_0^{2\pi} N\left(r, \frac{1}{f - e^{i\varphi}}\right) d\varphi - N(r, f) + \log^+ |f(0)|. \end{aligned}$$

The order of integration on the left hand side can be interchanged, by Fubini's theorem. Here the measurability of

$$(\varphi, \theta) \mapsto \log |f(re^{i\theta}) - e^{i\varphi}|$$

is clear; one has to show the existence of

$$\int_0^{2\pi} \int_0^{2\pi} \left| \log \left| f(re^{i\theta}) - e^{i\varphi} \right| \right| d\theta d\varphi.$$

We omit this here.

Interchanging the order of integration and using Lemma 5.1 again we obtain

$$\frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f(re^{i\theta}) \right| d\theta = \frac{1}{2\pi} \int_0^{2\pi} N\left(r, \frac{1}{f - e^{i\varphi}}\right) d\varphi - N(r, f) + \log^+ |f(0)|,$$

and hence the conclusion. □

Remark. (1) If $f(0) = \infty$, then the above proof gives

$$T(r, f) = \frac{1}{2\pi} \int_0^{2\pi} N\left(r, \frac{1}{f - e^{i\varphi}}\right) d\varphi + \log |c_\ell|$$

with the first non-zero Laurent coefficient c_ℓ.

(2) For $\varphi \in [0, 2\pi]$ with $f(0) \neq e^{i\varphi}$ we have

$$N\left(r, \frac{1}{f - e^{i\varphi}}\right) = \int_0^r n\left(t, \frac{1}{f - e^{i\varphi}}\right) \frac{dt}{t}.$$

Interchanging the order of integration as before we obtain with

$$s(t,f) = \frac{1}{2\pi}\int_0^{2\pi} n\left(t, \frac{1}{f-e^{i\varphi}}\right) d\varphi$$

that

$$T(r,f) = \int_0^r \frac{s(t,f)}{t} dt + \log^+ |f(0)|.$$

We recall that for an interval I a function $\Phi\colon I \to \mathbb{R}$ is called *convex* if for all $x, x_1, x_2 \in I$ with $x_1 \le x \le x_2$ we have

$$\Phi(x) \le \frac{x - x_1}{x_2 - x_1}\Phi(x_2) + \frac{x_2 - x}{x_2 - x_1}\Phi(x_1).$$

Equivalently,

$$\frac{\Phi(x) - \Phi(x_1)}{x - x_1} \le \frac{\Phi(x_2) - \Phi(x)}{x_2 - x}.$$

A differentiable function Φ is convex if and only if Φ' is non-decreasing. In fact, if $\Psi\colon I \to \mathbb{R}$ is non-decreasing and $x_0 \in I$, then $\Phi\colon I \to \mathbb{R}$,

$$\Phi(x) = \int_{x_0}^x \Psi(t) dt$$

is convex. (For continuous Ψ we of course have $\Phi' = \Psi$.)

For an interval $I \subset (0,\infty)$ and $\Phi\colon I \to \mathbb{R}$ we say that $\Phi(r)$ *is convex in* $\log r$ if the function $\Phi \circ \exp\colon \log(I) \to \mathbb{R}$ is convex. This means that Φ is convex on a logarithmic scale.

Analogously to the above, if $\Psi\colon I \to \mathbb{R}$ is non-decreasing, then $\Phi\colon I \to \mathbb{R}$,

$$\Phi(r) = \int_{r_0}^r \frac{\Psi(t)}{t} dt$$

is convex in $\log r$, for any fixed $r_0 \in I$. In fact, with $x_0 = \log r_0$ we have

$$\Phi(e^x) = \int_{r_0}^{e^x} \frac{\Psi(t)}{t} dt = \int_{x_0}^x \Psi(e^u) du.$$

We conclude from the above:

Theorem 5.5. *Let f be meromorphic. Then $N(r,f)$ and $T(r,f)$ are non-decreasing and convex in* $\log r$.

Remark. In general, $m(r,f)$ is neither convex in $\log r$ nor increasing.

6 Maximum modulus and characteristic of entire functions

Our starting point for Jensen's formula (Theorem 3.1) was the mean value property

$$f(0) = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) d\theta = \frac{1}{2\pi i} \int_{|z|=r} f(\zeta) \frac{d\zeta}{\zeta}$$

for a function f holomorphic in $\overline{D}(0, r)$, and the corresponding formula with f replaced by $\operatorname{Re} f$.

By Cauchy's integral formula we have

$$f(z) = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(\zeta)}{\zeta - z} d\zeta,$$

but taking real parts does not give a formula for $\operatorname{Re} f(z)$ in terms of $\operatorname{Re} f(re^{i\theta})$. Instead we use the following formula.

Theorem 6.1. *Let f be holomorphic in $\overline{D}(0, r)$ and $z \in D(0, r)$. Then*

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) K(z, r, \theta) d\theta$$

where

$$K(z, r, \theta) = \frac{r^2 - |z|^2}{|re^{i\theta} - z|^2}.$$

Remark. The expression $K(z, r, \theta)$ is called *Poisson kernel*, the formula is called *Poisson integral formula.*

Usually the Poisson integral formula is stated for harmonic functions. We recall the definition. For a domain D a function $u \colon D \to \mathbb{R}$ (or $u \colon D \to \mathbb{C}$) is called *harmonic* if it is twice continuously differentiable and

$$\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

If f is holomorphic, then $\operatorname{Re} f$ is harmonic. This is an immediate consequence of the Cauchy-Riemann equations. In the opposite direction, if $u \colon D \to \mathbb{R}$ is harmonic and D is simply connected, then there exists a holomorphic function f with $u = \operatorname{Re} f$. This fails if the domain is not simply connected, an example being $u \colon \mathbb{C} \setminus \{0\} \to \mathbb{R}$, $u(z) = \log |z|$. But in any simply connected subdomain D of $\mathbb{C} \setminus \{0\}$ we may define a branch $\arg \colon D \to \mathbb{R}$ of the argument, and $\log z = \log |z| + i \arg z$ defines a holomorphic function in D. Thus the theory of harmonic functions is closely connected to that of holomorphic functions. We will not consider harmonic functions in detail and thus have stated Theorem 6.1 for holomorphic functions. The above considerations show, however, that it remains valid for harmonic functions, and in fact the statement for harmonic functions is equivalent to that for holomorphic functions.

Proof of Theorem 6.1. The function

$$h\colon \overline{D}(0,r) \to \overline{D}(0,r), \quad h(\zeta) = \frac{r^2(\zeta + z)}{\overline{z}\zeta + r^2}$$

is bijective, with inverse function

$$k(\zeta) := h^{-1}(\zeta) = -r^2 \frac{\zeta - z}{\overline{z}\zeta - r^2},$$

and we have $h(\partial D(0,r)) = \partial D(0,r)$; see Lemma 3.1 where $h(\zeta)/r$ is considered. We apply the mean value property (Theorem 3.1) to $g = f \circ h$ and obtain

$$\begin{aligned} f(z) = f(h(0)) &= g(0) \\ &= \frac{1}{2\pi i} \int_{|w|=r} \frac{g(w)}{w} dw \\ &= \frac{1}{2\pi i} \int_{|w|=r} \frac{f(h(w))}{w} dw \\ &= \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f(\zeta)}{k(\zeta)} k'(\zeta) d\zeta \\ &= \frac{1}{2\pi i} \int_{|\zeta|=r} f(\zeta) \left(\frac{1}{\zeta - z} - \frac{\overline{z}}{\overline{z}\zeta - r^2} \right) d\zeta \end{aligned}$$

Now, for $|\zeta| = r$,

$$\begin{aligned} \frac{1}{\zeta - z} - \frac{\overline{z}}{\overline{z}\zeta - r^2} &= \frac{1}{\zeta - z} - \frac{\overline{z}}{(\overline{z} - \overline{\zeta})\zeta} \\ &= \frac{(\overline{z} - \overline{\zeta})\zeta - \overline{z}(\zeta - z)}{(\zeta - z)(\overline{z} - \overline{\zeta})\zeta} \\ &= \frac{\overline{\zeta}\zeta - \overline{z}z}{(\zeta - z)(\overline{z} - \overline{\zeta})\zeta} \\ &= \frac{r^2 - |z|^2}{|\zeta - z|^2} \cdot \frac{1}{\zeta} \end{aligned}$$

and thus

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta|=r} f(\zeta) \frac{r^2 - |z|^2}{|\zeta - z|^2} \frac{d\zeta}{\zeta} = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) \frac{r^2 - |z|^2}{|re^{i\theta} - z|^2} d\theta. \qquad \square$$

Corollary. Let f be holomorphic in $\overline{D}(0,r)$ and $z \in D(0,r)$. Then

$$\operatorname{Re} f(z) = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} f(re^{i\theta}) K(z,r,\theta) d\theta.$$

Jensen's formula was obtained from the mean value property. Applying similar arguments to the Poisson integral formula, we obtain the following result.

Theorem 6.2 (Poisson-Jensen-Nevanlinna formula)**.** *Let f be meromorphic in $\overline{D}(0,r)$, with zeros $a_1,\dots,a_m$ and poles $b_1,\dots,b_n$ in $D(0,r)$. Let $z \in D(0,r)$ with $f(z) \neq 0,\infty$. Then*

$$\log|f(z)| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(re^{i\theta}\right)\right| K(z,r,\theta)d\theta + \sum_{k=1}^{n} \log\left|\frac{r^2-\overline{b}_k z}{r(z-b_k)}\right| - \sum_{j=1}^{m} \log\left|\frac{r^2-\overline{a}_j z}{r(z-a_j)}\right|.$$

Remark. Jensen's formula is the special case $z = 0$.

Proof. We restrict, as in the proof of Jensen's formula (Theorem 3.2) to the case that f has no zeros and poles on $\partial D(0,r)$, and we proceed as there.

With

$$\varphi_a(z) = \frac{r(z-a)}{r^2-\overline{a}z}$$

we consider

$$h(z) = f(z)\frac{\prod_{k=1}^{n}\varphi_{b_k}(z)}{\prod_{j=1}^{m}\varphi_{a_j}(z)}.$$

Then h is holomorphic in $\overline{D}(0,r)$ and has no zeros there. Thus $\log|h|$ is the real part of a function holomorphic in $\overline{D}(0,r)$ and we have $|h(\zeta)| = |f(\zeta)|$ for $\zeta \in D(0,r)$.

Hence

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} \log\left|f(re^{i\theta})\right| K(z,r,\theta)d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log\left|h(re^{i\theta})\right| K(z,r,\theta)d\theta \\
&= \log|h(z)| \\
&= \log|f(z)| + \sum_{k=1}^{n}\log|\varphi_{b_k}(z)| - \sum_{j=1}^{m}\log|\varphi_{a_j}(z)| \\
&= \log|f(z)| - \sum_{k=1}^{n}\log\left|\frac{r^2-\overline{b}_k z}{r(z-b_k)}\right| + \sum_{j=1}^{m}\log\left|\frac{r^2-\overline{a}_j z}{r(z-a_j)}\right|
\end{aligned}$$

as claimed. □

Theorem 6.3. *Let f be entire and $0 < r < R$. Then*

$$T(r,f) \leq \log^+ M(r,f) \leq \frac{R+r}{R-r}T(R,f).$$

Proof. The first inequality is obvious since

$$T(r,f) = m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+\left|f\left(re^{i\theta}\right)\right| d\theta.$$

To prove the second one, let $z \in \mathbb{C}$ with $|z| = r$. By the Poisson-Jensen-Nevanlinna formula,

$$\log|f(z)| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(Re^{i\theta}\right)\right| K(z,R,\theta)d\theta - \sum_{j=1}^{m} \log\left|\frac{R^2 - \overline{a}_j z}{R(z-a_j)}\right|.$$

Using the functions φ_{a_j} from the previous proof, that is,

$$\varphi_{a_j}(z) = \frac{R(z-a_j)}{R^2 - \overline{a}_j z},$$

we have $\varphi_{a_j}(D(0,R)) = D(0,1)$ and thus

$$-\log\left|\frac{R^2 - \overline{a}_j z}{R(z-a_j)}\right| = \log|\varphi_{a_j}(z)| \leq 0.$$

We also have

$$0 \leq K(z,R,\theta) = \frac{R^2 - |z|^2}{|Re^{i\theta} - z|^2} \leq \frac{R^2 - |z|^2}{(R - |z|)^2} = \frac{R + |z|}{R - |z|} = \frac{R+r}{R-r}.$$

It follows that

$$\begin{aligned}
\log|f(z)| &\leq \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(Re^{i\theta}\right)\right| K(z,R,\theta)d\theta \\
&\leq \frac{1}{2\pi}\int_0^{2\pi} \log^+\left|f\left(Re^{i\theta}\right)\right| K(z,R,\theta)d\theta \\
&\leq \frac{1}{2\pi}\int_0^{2\pi} \log^+\left|f\left(Re^{i\theta}\right)\right| \frac{R+r}{R-r}d\theta \\
&= \frac{R+r}{R-r}T(R,f).
\end{aligned}$$

□

The following result is a generalization of Liouville's theorem that a bounded entire function is constant.

Theorem 6.4. *Let f be an entire function and suppose that*

$$L := \liminf_{r\to\infty} \frac{\log M(r,f)}{\log r} < \infty.$$

Then f is a polynomial and $\deg(f) \leq L$.

Proof. Let

$$f(z) = \sum_{n=1}^{\infty} c_n z^n$$

be the Taylor series expansion of f. Then

$$c_n = \frac{1}{2\pi i}\int_{|z|=r} \frac{f(z)}{z^{n+1}}dz$$

and thus

$$|c_n| \leq \frac{1}{2\pi} \cdot 2\pi r \frac{M(r,f)}{r^{n+1}} = \frac{M(r,f)}{r^n}$$

for all $r > 0$ and all $n \in \mathbb{N}$. (The last inequality is known as Cauchy's inequality.)

Given $\varepsilon > 0$ there exist arbitrarily large r with

$$\frac{\log M(r,f)}{\log r} \leq L + \varepsilon$$

and thus

$$M(r,f) \leq r^{L+\varepsilon}.$$

It follows that

$$|c_n| \leq r^{L+\varepsilon-n}$$

for arbitrarily large r and thus that $c_n = 0$ for $n > L+\varepsilon$. The conclusion follows. □

The following result is an analogue of Theorem 6.4 for meromorphic functions.

Theorem 6.5. *Let f be meromorphic and suppose that*

$$L := \liminf_{r\to\infty} \frac{T(r,f)}{\log r} < \infty.$$

Then f is a rational function (and $\deg(f) = L$*).*

Proof. Let $C > 1$. By Lemma 3.3, applied with $R = r^C$, we have

$$\begin{aligned} n(r,f) &\leq \frac{1}{\log(r^C/r)} \cdot N(r^C,f) \\ &= \frac{C}{C-1} \cdot \frac{N(r^C,f)}{\log r^C} \\ &\leq \frac{C}{C-1} \cdot \frac{T(r^C,f)}{\log r^C} \\ &\leq \frac{C}{C-1}(L+\varepsilon) \end{aligned}$$

for arbitrarily large r, for any given $\varepsilon > 0$. Thus f has at most L poles. Hence there exists a polynomial P (of degree at most L) such that $g := Pf$ is entire. Since $T(r,P) \sim \deg(P)\log r$ as $r \to \infty$ (by Example 2 after Theorem 4.1) we find, using Theorem 5.1, that

$$T(r,g) \leq T(r,P) + T(r,f) \leq (2L+1)\log r$$

for arbitrarily large r. Applying Theorem 6.3 with $R = 2r$ we thus find that

$$\log M(r,g) \leq 3T(2r,g) \leq 3(2L+1)\log(2r) \leq 3(2L+2)\log r$$

for arbitrarily large r. Theorem 6.4 yields that g is a polynomial. Hence f is rational. Using that $T(r,f) \sim \deg(f)\log r$ as $r \to \infty$ we find that $\deg(f) = L$. □

A meromorphic function which is not rational is called *transcendental*. Theorem 6.5 says that

$$\lim_{r\to\infty} \frac{T(r,f)}{\log r} = \infty$$

for a transcendental meromorphic function f.

Applying Theorem 6.3 with $R = Kr$ where $K > 1$ we obtain

$$\log M(r,f) \le \frac{K+1}{K-1} T(Kr,f).$$

(For $K = 2$ we used this already in the above proof.) If

$$T(r,f) \le \tau r^\rho$$

for large r, with $\tau, \rho > 0$, then

$$\log M(r,f) \le \frac{K+1}{K-1} K^\rho \tau r^\rho.$$

Choosing $K = 1 + 1/\rho$ we thus have

$$\log M(r,f) \le \left(2 + \frac{1}{\rho}\right) \rho \left(1 + \frac{1}{\rho}\right)^\rho \tau r^\rho \le \left(2 + \frac{1}{\rho}\right) \rho e \tau r^\rho.$$

So up to the constant $(2 + 1/\rho)\,\rho e$ the maximum modulus does not grow faster. The constant $(2 + 1/\rho)\,\rho e$ is not optimal. For $\rho > \frac{1}{2}$ the sharp constant is $\pi\rho$.

Theorem 6.3 compares $\log M(r,f)$ with $T(R,f)$ for some $R > r$. The following lemma will allow to compare $\log M(r,f)$ with $T(r,f)$, but not for all values of r.

Lemma 6.1 (Borel)**.** *Let $x_0, y_0 > 0$, $\delta > 0$, $u\colon [x_0,\infty) \to [y_0,\infty)$ continuous and non-decreasing and $\varphi\colon [y_0,\infty) \to (0,\infty)$ continuous and non-increasing. Suppose that*

$$\int_{y_0}^\infty \varphi(x)dx < \infty.$$

Put

$$E = \{x \in [x_0,\infty)\colon u(x + \varphi(u(x))) \ge u(x) + \delta\}.$$

Then there exist sequences (x_k) and (x_k') in $[x_0,\infty)$, with $x_k < x_k' \le x_{k+1}$ for all $k \in \mathbb{N}$, such that

$$E \subset \bigcup_{k=1}^\infty [x_k, x_k']$$

and

$$\sum_{k=1}^\infty (x_k' - x_k) < \infty.$$

Remark. The last two conditions imply that E has finite measure. The lemma says that if $x \notin E$, then $u(x + \varphi(u(x))) < u(x) + \delta$.

Proof. We may assume that E is unbounded, since otherwise the result is obvious.

Put $x_1 = \min E$. Note that this minimum exists, since u and φ are continuous and hence E is closed. Put $x_1' = x_1 + \varphi(u(x_1))$.

We define (x_k) and (x_k') recursively by $x_k = \min\bigl(E \cap [x_{k-1}', \infty)\bigr)$ and $x_k' = x_k + \varphi(u(x_k))$. By construction, $x_k < x_k' \le x_{k+1}$ and

$$u(x_{k+1}) \ge u(x_k') = u(x_k + \varphi(u(x_k))) \ge u(x_k) + \delta$$

for all $k \in \mathbb{N}$. We deduce that

$$u(x_{k+1}) \ge u(x_1) + k\delta$$

and thus $u(x_k) \to \infty$. Hence $x_k \to \infty$.

By construction we thus have

$$E \subset \bigcup_{k=1}^{\infty} [x_k, x_k'] .$$

Finally, if $k \ge 2$, then

$$\begin{aligned} x_k' - x_k &= \varphi(u(x_k)) \\ &\le \varphi(u(x_1) + (k-1)\delta) \\ &\le \varphi(y_0 + (k-1)\delta) \\ &= \varphi(y_0 + (k-1)\delta) \cdot \frac{1}{\delta} \int_{y_0+(k-2)\delta}^{y_0+(k-1)\delta} dx \\ &\le \frac{1}{\delta} \int_{y_0+(k-2)\delta}^{y_0+(k-1)\delta} \varphi(x) dx \end{aligned}$$

and thus

$$\sum_{k=2}^{\infty} (x_k' - x_k) \le \frac{1}{\delta} \int_{y_0}^{\infty} \varphi(x) dx < \infty. \qquad \square$$

Remark. The result is often applied to the function $v = e^u$. With $K = e^\delta > 1$ we obtain

$$v(x + \varphi(\log v(x))) < Kv(x) \quad \text{for } x \notin E.$$

For example, choosing $\varphi(x) = e^{-x}$ we find that if v is increasing and continuous, then

$$v\left(x + \frac{1}{v(x)}\right) < Kv(x) \quad \text{for } x \notin E.$$

In this form it is found in most standard books. Sometimes the result is also applied to $w = v \circ \log$. Using that $1 + y \le e^y$ for $y \in \mathbb{R}$ we find that

$$\begin{aligned} w(r(1 + \varphi(\log w(r)))) &\le w\bigl(re^{\varphi(\log w(r))}\bigr) \\ &= v(\log r + \varphi(\log v(\log r))) \\ &\le Kv(\log r) \\ &= Kw(r) \end{aligned}$$

if $\log r \notin E$.

Let
$$F = \{r \colon \log r \in E\}.$$
Putting $r_k = e^{x_k}$ and $r'_k = e^{x'_k}$ the condition
$$\sum_{k=1}^{\infty} (x'_k - x_k) < \infty$$
takes the form
$$\sum_{k=1}^{\infty} \log \frac{r'_k}{r_k} < \infty,$$
and we have
$$F \subset \bigcup_{k=1}^{\infty} [r_k, r'_k].$$
In particular, we have
$$\int_F \frac{dt}{t} < \infty,$$
corresponding to the condition
$$\int_E dx < \infty.$$
We say that F has *finite logarithmic measure.*

Theorem 6.6. *Let f be entire, $r_0 > 0$ and $\varphi \colon [r_0, \infty) \to \mathbb{R}$ continuous and decreasing with*
$$\int_{r_0}^{\infty} \varphi(t) dt < \infty.$$
Then there exists a (closed) set $F \subset [r_0, \infty)$ with
$$\int_F \frac{dt}{t} < \infty$$
such that
$$\log M(r, f) \le \frac{T(r, f)}{\varphi(\log T(r, f))} \quad \text{for } r \notin F.$$

Remark. Taking $\varphi(t) = t^{-\alpha}$ where $\alpha > 1$ we obtain
$$\log M(r, f) \le T(r, f) [\log T(r, f)]^{\alpha} \quad \text{for } r \notin F.$$

Proof of Theorem 6.6. We apply Borel's Lemma 6.1 and the remark following its proof with $w(r) = T(r, f)$ and $K = 2$. Taking $R = r(1 + \varphi(\log T(r, f)))$ in

Theorem 6.3 we have

$$\begin{aligned}\log M(r,f) &\leq \frac{R+r}{R-r}T(R,f)\\ &\leq \frac{2+\varphi(\log T(r,f))}{\varphi(\log T(r,f))}\cdot 2T(r,f)\\ &\leq 5\frac{T(r,f)}{\varphi(\log T(r,f))}\end{aligned}$$

for large $r \notin F$. (Here we have used that $\int_{r_0}^{\infty}\varphi(t)dt < \infty$ and the monotonicity of φ implies that $\lim_{t\to\infty}\varphi(t) = 0$.)

Since the hypothesis remains valid if φ is replaced by $\frac{1}{5}\varphi$, the conclusion follows. □

Theorem 5.5 says that $T(r,f)$ and $N(r,f)$ are convex in $\log r$ (and non-decreasing). The following result is an analogue for the maximum modulus.

Theorem 6.7 (Hadamard three circle theorem)**.** *Let f be entire, $f \not\equiv 0$. Then $\log M(r,f)$ is convex in $\log r$.*

Proof. We have to show that if $0 < r_1 < r < r_2$, then

$$\log M(r,f) \leq \frac{\log r_2 - \log r}{\log r_2 - \log r_1}\log M(r_1,f) + \frac{\log r - \log r_1}{\log r_2 - \log r_1}\log M(r_2,f).$$

In order to do so, we consider for $\alpha \in \mathbb{R}$ the function

$$\phi\colon\ \mathbb{C}\setminus\{0\} \to \mathbb{R},\ \ \phi(z) = |z|^{\alpha}|f(z)|.$$

Then ϕ has no strict local maxima, since locally ϕ is the absolute value of a holomorphic function. Indeed, in any simply connected subdomain U of $\mathbb{C}\setminus\{0\}$ there exists a branch log of the logarithm, and

$$\phi(z) = |z^{\alpha}f(z)| = |e^{\alpha\log z}f(z)|$$

for $z \in U$.

We now choose $\alpha \in \mathbb{R}$ such that

$$\max_{|z|=r_1}\phi(z) = \max_{|z|=r_2}\phi(z),$$

which is equivalent to

$$r_1^{\alpha}M(r_1,f) = r_2^{\alpha}M(r_2,f).$$

Solving the equation for α shows that this holds for

$$\alpha = -\frac{\log M(r_2,f) - \log M(r_1,f)}{\log r_2 - \log r_1}.$$

Since ϕ has no local maxima we have

$$r^{\alpha}M(r,f) \leq r_1^{\alpha}M(r_1,f) = r_1^{\alpha}M(r_2,f).$$

Inserting the value of α found in this equation yields the conclusion. □

Instead of

$$M(r,f) = \max_{|z|=r} |f(z)|$$

we will sometimes consider

$$A(r,f) := \max_{|z|=r} \operatorname{Re} f(z).$$

To relate the two quantities we will use the following theorem.

Theorem 6.8. *Let f be holomorphic in $\overline{D}(0,r)$ and $z \in D(0,r)$. Then*

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re} f(re^{i\theta}) \frac{re^{i\theta}+z}{re^{i\theta}-z} d\theta + i \operatorname{Im} f(0).$$

Proof. Putting $\xi = re^{i\theta}$ we have

$$\begin{aligned}
\operatorname{Re}\left(\frac{re^{i\theta}+z}{re^{i\theta}-z}\right) &= \operatorname{Re}\left(\frac{\xi+z}{\xi-z}\right) \\
&= \operatorname{Re}\left(\frac{(\xi+z)(\overline{\xi}-\overline{z})}{|\xi-z|^2}\right) \\
&= \operatorname{Re}\left(\frac{|\xi|^2 - \xi\overline{z} + z\overline{\xi} - |z|^2}{|\xi-z|^2}\right) \\
&= \operatorname{Re}\left(\frac{|\xi|^2 + 2i\operatorname{Im}(z\overline{\xi}) - |z|^2}{|\xi-z|^2}\right) \\
&= \frac{|\xi|^2 - |z|^2}{|\xi-z|^2} \\
&= K(z,r,\theta).
\end{aligned}$$

Putting

$$h(z) = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re} f(re^{i\theta}) \frac{re^{i\theta}-z}{re^{i\theta}+z} d\theta$$

we find that

$$\operatorname{Re} h(z) = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re} f(re^{i\theta}) K(z,r,\theta) d\theta = \operatorname{Re} f(z)$$

by Poisson's integral formula (Theorem 6.1 and the corollary of it).

The above expression for h defines a holomorphic function $h\colon D(0,r) \to \mathbb{C}$. We deduce that $f = h + c$ with a constant c. The constant c is then computed by considering the value for $z = 0$. We conclude that

$$c = f(0) - h(0) = f(0) - \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re} f(re^{i\theta}) d\theta = f(0) - \operatorname{Re} f(0) = i \operatorname{Im} f(0)$$

and the conclusion follows. ∎

Theorem 6.9. *Let f be entire and $0 < r < R$. Then*

$$M(r, f) \le 2\frac{R+r}{R-r}\left(\max\{A(R, f), 0\} + |f(0)|\right).$$

Proof. Let $z \in \mathbb{C}$ with $|z| = r$. Theorem 6.7 yields that

$$\begin{aligned} |f(z)| &= \left|\frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re} f(Re^{i\theta})\frac{Re^{i\theta}+z}{Re^{i\theta}-z}d\theta + i\operatorname{Im} f(0)\right| \\ &\le \frac{1}{2\pi}\int_0^{2\pi} \left|\operatorname{Re} f(Re^{i\theta})\right| \frac{R+|z|}{R-|z|}d\theta + |\operatorname{Im} f(0)| \\ &= \frac{R+r}{R-r}\cdot\frac{1}{2\pi}\int_0^{2\pi} \left|\operatorname{Re} f(Re^{i\theta})\right| d\theta + |\operatorname{Im} f(0)|. \end{aligned}$$

We also have

$$\operatorname{Re} f(0) = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re} f(Re^{i\theta})d\theta$$

and thus

$$\frac{R+r}{R-r}\operatorname{Re} f(0) = \frac{R+r}{R-r}\cdot\frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re} f(Re^{i\theta})d\theta.$$

Adding this to the first inequality and noting that $|x| + x = 2\max\{x, 0\}$ for $x \in \mathbb{R}$ we obtain

$$\begin{aligned} |f(z)| + \frac{R+r}{R-r}\operatorname{Re} f(0) &\le \frac{R+r}{R-r}\cdot\frac{1}{2\pi}\int_0^{2\pi} 2\max\{\operatorname{Re} f(Re^{i\theta}), 0\}d\theta + |\operatorname{Im} f(0)| \\ &\le 2\frac{R+r}{R-r}\max\{A(R, f), 0\} + |\operatorname{Im} f(0)|. \end{aligned}$$

We conclude that

$$\begin{aligned} M(r, f) &\le 2\frac{R+r}{R-r}\max\{A(R, f), 0\} + |\operatorname{Im} f(0)| - \frac{R+r}{R-r}\operatorname{Re} f(0) \\ &\le 2\frac{R+r}{R-r}\left(\max\{A(R, f), 0\} + |f(0)|\right). \end{aligned}$$ □

Remark. If f is a non-constant entire function, then $M(r, f) \to \infty$ by Liouville's theorem. Theorem 6.9 implies that also $A(r, f) \to \infty$. In particular, $A(r, f) > 0$ for large r so that we actually have

$$M(r, f) \le 2\frac{R+r}{R-r}\left(A(R, f) + |f(0)|\right).$$

for $0 < r < R$ if R is large.

In analogy to Theorem 6.4 and Theorem 6.5 we have the following result.

Theorem 6.10. *Let f be entire with*

$$L := \liminf_{r\to\infty} \frac{\log A(r, f)}{\log r} < \infty.$$

Then f is a polynomial with $\deg(f) \le L$.

The proof is a direct consequence of Theorem 6.4 and Theorem 6.9.

7 The order of a meromorphic function

Definition 7.1. Let f be meromorphic. Then

$$\rho(f) := \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}$$

is called the *order* of f and

$$\lambda(f) := \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r}$$

is called the *lower order* of f.

Example. Since $T(r,\exp) = \dfrac{r}{\pi}$, we have $\rho(\exp) = \lambda(\exp) = 1$.

Remark. Suppose that $T(r,f) \le r^K$ for $r \ge r_0$. Then

$$\frac{\log T(r,f)}{\log r} \le K \quad \text{for} \quad r \ge r_0$$

and thus $\rho(f) \le K$. The order is the infimum of the set of all K for which an estimate $T(r,f) \le r^K$ holds for all large r. An analogous remark applies to the lower order.

Theorem 7.1. *Let f be entire. Then*

$$\rho(f) = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r}$$

and

$$\lambda(f) = \liminf_{r\to\infty} \frac{\log\log M(r,f)}{\log r}.$$

The proof follows easily from Theorem 6.4 and is omitted here.

Example. (1) Let

$$f(z) = e^{e^z}.$$

Then $M(r,f) = e^{e^r}$, so $\log\log M(r,f) = r$ and hence $\rho(f) = \lambda(f) = \infty$.

(2) Let

$$f(z) = \cos z = \frac{e^{iz} + e^{-iz}}{2} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k}.$$

Then

$$M(r,f) \le \frac{e^r + e^r}{2} = e^r$$

and

$$M(r,f) \ge |f(ir)| = \frac{e^r + e^{-r}}{2} \ge \frac{e^r}{2}.$$

It follows that $\log M(r,f) = r + \mathcal{O}(1)$ and hence $\rho(f) = \lambda(f) = 1$.

(3) Let

$$f(z) = \cos\sqrt{z} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^k.$$

Then

$$M(r,f) \le e^{\sqrt{r}} \quad \text{and} \quad M(r,f) \ge f(-r) \ge \frac{e^{\sqrt{r}}}{2}$$

and hence $\rho(f) = \lambda(f) = \frac{1}{2}$.

The following result is classical, but we omit the proof.

Theorem 7.2. *Let f be entire with Taylor series expansion*

$$f(z) = \sum_{n=0}^{\infty} c_n z^n.$$

Then

$$\rho(f) = \limsup_{n\to\infty} \frac{n\log n}{-\log|c_n|}.$$

Here we put $n\log n/(-\log|c_n|) = 0$ if $c_n = 0$.

Remark. (1) The result gives an easy possibility to construct functions with $\rho(f) = \mu$ for any preassigned $\mu \ge 1$. E.g. for $0 < \mu < \infty$ we can take $c_n = n^{-n/\rho}$.

(2) Taking the lower limit in the formula in Theorem 7.2 does in general *not* give the lower order. However, we have

$$\lambda(f) = \max_{(n_k)} \liminf_{k\to\infty} \frac{n_k \log n_{k-1}}{-\log|c_{n_k}|},$$

with the maximum taken over all increasing sequences (n_k) in $\mathbb{N}$.

Theorem 7.3. *Let f be meromorphic and M a Möbius transformation. Then $\rho(M\circ f) = \rho(f)$ and $\lambda(M\circ f) = \lambda(f)$.*

The proof follows directly from Theorem 5.2.

Theorem 7.4. *Let f_1 and f_2 be meromorphic. Then*

$$\rho(f_1 \pm f_2) \le \max\{\rho(f_1), \rho(f_2)\},$$

$$\rho(f_1 \cdot f_2) \le \max\{\rho(f_1), \rho(f_2)\},$$

and

$$\rho(f_1/f_2) \le \max\{\rho(f_1), \rho(f_2)\}.$$

If $\rho(f_1) \ne \rho(f_2)$, then we have equality in the above estimates.

Proof. We only consider the sum $f = f_1 + f_2$, the other cases are analogous.

By Theorem 5.1 we have

$$T(r,f) \le T(r,f_1) + T(r,f_2) + \log 2 \le 2\max\{T(r,f_1), T(r,f_2)\} + \log 2$$

and thus

$$\log T(r,f) \le \log^+ \max\{T(r,f_1), T(r,f_2)\} + \mathcal{O}(1)$$

which implies that $\rho(f) \le \max\{\rho(f_1), \rho(f_2)\}$. Suppose that $\rho(f_1) \ne \rho(f_2)$, say $\rho(f_1) < \rho(f_2)$. Then

$$\rho(f_2) = \rho(f - f_1) \le \max\{\rho(f_1), \rho(f_2)\} \le \rho(f)$$

since $\rho(f_1) < \rho(f_2)$. Thus $\rho(f) = \rho(f_2) = \max\{\rho(f_1), \rho(f_2)\}$. □

8 Weierstraß products

The fundamental theorem of algebra says that a polynomial $p(z) = a_d z^d + \cdots + a_0$ with $a_0, a_d \ne 0$ has a factorization

$$p(z) = a_d \prod_{j=1}^{d} (z - z_j) = a_0 \prod_{j=1}^{d} \left(1 - \frac{z}{z_j}\right),$$

with the zeros $z_1, \ldots, z_d$ of p.

We consider the question whether entire functions with infinitely many zeros can be written as infinite products in a similar way.

Let $(a_j)_{j\in\mathbb{N}}$ be a sequence of complex number. A naive definition for the convergence of the infinite product $\prod_{j=1}^{\infty} a_j$ would be the existence of $\lim_{k\to\infty} \prod_{j=1}^{k} a_j$. This definition would have two disadvantages:

- if $a_n = 0$ for some n, then $\lim_{k\to\infty} \prod_{j=1}^{k} a_j = 0$, regardless of the behaviour of a_j as $j \to \infty$;
- $\lim_{k\to\infty} \prod_{j=1}^{k} a_j = 0$ is possible even if $a_j \ne 0$ for all j. For example, this happens for $a_j = j/(j+1)$.

Definition 8.1. Let (a_j) be a sequence in $\mathbb{C}$. Then $\prod_{j=1}^{\infty} a_j$ is called *convergent*, if there exists $N \in \mathbb{N}$ with $a_j \ne 0$ for $j \ge N$ and if $\lim_{k\to\infty} \prod_{j=N}^{k} a_j$ exists and $\lim_{k\to\infty} \prod_{j=N}^{k} a_j \ne 0$. In this case we put

$$\prod_{j=1}^{\infty} a_j := a_1 \cdot a_2 \cdot \ldots \cdot a_{N-1} \cdot \lim_{k\to\infty} \prod_{j=N}^{k} a_j.$$

For a convergent infinite product $\prod_{j=1}^{\infty} a_j$ we easily see that $\prod_{j=1}^{\infty} a_j = 0$ if and only if there exists $j \in \mathbb{N}$ with $a_j = 0$. For $k > N$ we have

$$a_k = \frac{\prod_{j=N}^{k} a_j}{\prod_{j=N}^{k-1} a_j}$$

and thus $a_k \to 1$ if the product $\prod_{j=1}^{\infty} a_j$ converges. This necessary condition for convergence is not sufficient, as shown by the example $a_j = j/(j+1)$ already considered.

The condition $a_k \to 1$ for infinite products corresponds to the condition $a_k \to 0$ for infinite series $\sum_{j=1}^{\infty} a_j$. The analogue of the Cauchy criterion for infinite series says that the infinite product $\prod_{j=1}^{\infty} a_j$ converges if and only if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\left| \prod_{j=m}^{n} a_j - 1 \right| < \varepsilon \quad \text{for } n > m \geq N.$$

Since $a_j \to 0$ for a convergent infinite product $\prod_{j=1}^{\infty} a_j$ we write the factors a_j in the form $a_j = 1 + c_j$.

The product $\prod_{j=1}^{\infty}(1+c_j)$ is called *absolutely convergent* if $\prod_{j=1}^{\infty}(1+|c_j|)$ converges. Since

$$\left| \prod_{j=m}^{n} (1+c_j) - 1 \right| \leq \prod_{j=m}^{n} (1+|c_j|) - 1$$

for $n \geq m$ we see that absolutely convergent infinite products are convergent. Moreover, the sequence

$$\left(\prod_{j=1}^{n} (1+|c_j|) \right)_{n \in \mathbb{N}}$$

is non-decreasing, so it converges if and only if it is bounded.

Theorem 8.1. *An infinite product $\prod_{j=1}^{n}(1+c_j)$ converges absolutely if and only if the series $\sum_{j=1}^{\infty} c_j$ converges absolutely.*

Proof. Since $\log(1+x) \leq x$ for $x > -1$ we have

$$\log\left(\prod_{j=1}^{n} (1+|c_j|) \right) = \sum_{j=1}^{n} \log(1+|c_j|) \leq \sum_{j=1}^{n} |c_j|$$

so that the absolute convergence of the series implies that of the infinite product. Suppose now that the infinite product converges absolutely. Then $c_j \to 0$ and thus there exists N with $|c_j| \leq 1$ for $j \geq N$. Since $x \leq 2\log(1+x)$ for $0 \leq x \leq 1$ we deduce that

$$\sum_{j=N}^{n} |c_j| \leq 2 \sum_{j=N}^{n} \log(1+|c_j|) = 2\log\left(\prod_{j=N}^{n} (1+|c_j|) \right)$$

for $n \geq N$, from which the conclusion follows. □

The above considerations extend to infinite products of functions. Definitions like (locally) uniform convergence can be generalized to infinite products in an obvious way. This yields the following result, the proof of which we omit.

Theorem 8.2. *Let D be a domain and let (u_j) be a sequence of functions holomorphic in D. Suppose that $\sum_{j=1}^{\infty} |u_j|$ converges locally uniformly in D. Then*

$$f(z) := \prod_{j=1}^{\infty}(1 + u_j(z))$$

converges locally uniformly in D and defines a holomorphic function $f\colon D \to \mathbb{C}$. Moreover, for $z \in D$ we have $f(z) = 0$ if and only if there exists $j \in \mathbb{N}$ such that $1 + u_j(z) = 0$.

When looking for an entire function whose zeros consist of a given sequence (z_j), it is tempting to choose $u_j(z) = -z/z_j$ and thus to consider the infinite product $\prod_{j=1}^{\infty}(1 - z/z_j)$. However, in general this infinite product will diverge. For example, this is the case for $z_j = j$. The following definition will lead to a suitable modification.

Definition 8.2. Let $q \in \mathbb{N}_0$. The entire functions $E(\cdot, q)$ given by

$$E(u, 0) = 1 - u$$

and

$$E(u, q) = (1 - u) \exp\left(\sum_{j=1}^{q} \frac{u^k}{k}\right)$$

for $q \in \mathbb{N}$ are called *Weierstraß primary factors.*

Lemma 8.1. *Let $q \in \mathbb{N}_0$ and $u \in \overline{D}(0, 1)$. Then*

$$|E(u, q) - 1| \leq |u|^{q+1}.$$

Proof. For $q = 0$ this is obvious. So let $q \in \mathbb{N}$. Then

$$\begin{aligned}
\frac{d}{du}E(u, q) &= -\exp\left(\sum_{k=1}^{q} \frac{u^k}{k}\right) + (1 - u) \exp\left(\sum_{k=1}^{q} \frac{u^k}{k}\right) \sum_{k=1}^{q} u^{k-1} \\
&= \exp\left(\sum_{k=1}^{q} \frac{u^k}{k}\right) \left(-1 + (1 - u) \sum_{k=0}^{q-1} u^k\right) \\
&= \exp\left(\sum_{k=1}^{q} \frac{u^k}{k}\right) \left(-1 + \sum_{k=0}^{q-1} \left(u^k - u^{k+1}\right)\right) \\
&= -u^q \cdot \exp\left(\sum_{k=1}^{q} \frac{u^k}{k}\right).
\end{aligned}$$

Since all coefficients of the Taylor series expansion of the right hand side are negative we conclude that

$$E(u, q) = 1 - \sum_{j=q+1}^{\infty} \alpha_j u^j$$

where $\alpha_j \geq 0$ for all $j \geq q+1$. Since $E(1,q) = 0$ we find that

$$\sum_{j=q+1}^{\infty} \alpha_j = 1.$$

Hence

$$\begin{aligned} |E(u,q) - 1| &\leq \sum_{j=q+1}^{\infty} \alpha_j |u|^j \\ &= |u|^{q+1} \sum_{j=q+1}^{\infty} \alpha_j |u|^{j-q-1} \\ &\leq |u|^{q+1} \sum_{j=q+1}^{\infty} \alpha_j \\ &= |u|^{q+1}. \end{aligned}$$

□

Theorem 8.3. *Let (z_j) be a sequence in $\mathbb{C} \setminus \{0\}$ with $\lim_{j\to\infty} |z_j| = \infty$. Then there exists a sequence (q_j) in $\mathbb{N}_0$ such that*

$$\sum_{j=1}^{\infty} \left| \frac{z}{z_j} \right|^{q_j+1}$$

converges locally uniformly in $\mathbb{C}$.

If (q_j) has this property, then

$$\prod_{j=1}^{\infty} E\left(\frac{z}{z_j}, q_j\right)$$

converges locally uniformly in $\mathbb{C}$ and represents an entire function whose zeros are precisely the z_j.

Here multiplicities are counted in the sense that if $a \in \mathbb{C}$ appears m times in the sequence (z_j), i.e., $\operatorname{card}\{j \in \mathbb{N} \colon z_j = a\} = m$, then a is a zero of multiplicity m.

Proof. The first claim is satisfied for $q_j = j$. To see this let $R > 0$. Then there exists $j_0 \in \mathbb{N}$ with $|z_j| \geq 2R$ for $j \geq j_0$. For $z \in D(0,R)$ and $j \geq j_0$ we thus have

$$\left| \frac{z}{z_j} \right|^{q_j+1} = \left| \frac{z}{z_j} \right|^{j+1} \leq \left(\frac{R}{2R} \right)^{j+1} = \frac{1}{2^{j+1}}.$$

This implies that the sequence $\sum_{j=1}^{\infty} |z/z_j|^{q_j+1}$ converges uniformly in $D(0,R)$. Since $R > 0$ was arbitrary this yields locally uniform convergence in $\mathbb{C}$.

Let now (q_j) be a sequence such that this series converges locally uniformly. Theorem 8.2 now yields that

$$\prod_{j=1}^{\infty} E\left(\frac{z}{z_j}, q\right)$$

converges locally uniformly and has the required properties. □

Remark. It is easy to show that $q_j = [\log j]$ is also an admissible choice. In Section 9 we will consider the case where (q_j) is constant.

Theorem 8.4 (Weierstraß factorization theorem)**.** *Let f be entire with infinitely many zeros. If $f(0) = 0$, let m be the multiplicity of this zero, and put $m = 0$ otherwise. Let (z_j) be the sequence of zeros of f in $\mathbb{C} \setminus \{0\}$, counted according to multiplicity, and let (q_j) be as in Theorem 8.3. Then there exists an entire function g such that*

$$f(z) = e^{g(z)} z^m \prod_{j=1}^{\infty} E\left(\frac{z}{z_j}, q_j\right).$$

Proof. The function

$$z \mapsto \frac{f(z)}{z^m \prod_{j=1}^{\infty} E\left(\frac{z}{z_j}, q_j\right)}$$

is entire without zeros and thus has the form e^g with an entire function g. □

Remark. If f has only finitely many zeros, we have the same result, except that the product occurring is finite.

Theorem 8.5. *Let f be meromorphic in $\mathbb{C}$. Then there are entire functions g and h such that $f = g/h$.*

Proof. Let (z_j) be the (finite or infinite) sequence of poles of f in $\mathbb{C} \setminus \{0\}$, counted according to multiplicity. By Theorem 8.3 there exists an entire function k which has precisely the z_j as zeros.

If 0 is a pole of f, let m be its multiplicity and put $m = 0$ otherwise. Define h by $h(z) = z^m k(z)$. Then h is entire and so is $g = fh$. The conclusion follows. □

9 The Hadamard factorization theorem

Definition 9.1. Let (z_j) be a sequence in $\mathbb{C} \setminus \{0\}$ with $|z_j| \to \infty$ as $j \to \infty$. Then

$$\sigma := \sigma((z_j)) := \inf \left\{ \mu \in \mathbb{R} \colon \sum_{j=1}^{\infty} \frac{1}{|z|^\mu} < \infty \right\}$$

is called the *exponent of convergence* of the sequence (z_j), with $\sigma = \infty$ if the series diverges for all μ.

If $\sigma < \infty$, then

$$q := q((z_j)) := \min \left\{ m \in \mathbb{N} \colon \sum_{j=1}^{\infty} \frac{1}{|z_j|^{m+1}} < \infty \right\}$$

is called the *genus* (German: *Geschlecht*) of the sequence (z_j).

Remark. 1. Clearly $q \leq \sigma \leq q + 1$.

2. The exponent of convergence is often denoted by λ in the literature, but we reserve this for the lower order.

3. If (z_j) is a sequence in $\mathbb{C}$ with $|z_j| \to \infty$ as $j \to \infty$, with $z_j = 0$ for some j, the exponent of convergence is defined by omitting these z_j from the sequence.

4. Any (infinite) discrete subset M of $\mathbb{C}$ has the form $M = \{z_j : j \in \mathbb{N}\}$ with some sequence (z_j) satisfying $|z_j| \to \infty$. We define the exponent of convergence of M as that of (z_j).

Theorem 9.1. *Let (z_j) be as in Definition 9.1. Let $n(r)$ be the number of z_j in $\overline{D}(0,r)$ and put*

$$N(r) = \int_0^r \frac{n(t)}{t} dt.$$

Then

$$\sigma = \limsup_{r\to\infty} \frac{\log n(r)}{\log r} = \limsup_{r\to\infty} \frac{\log N(r)}{\log r}.$$

In the proof we will use the following lemmas.

Lemma 9.1. *Let (z_j) and $n(r)$ be as in Theorem 9.1, and let $\mu > 0$. Then*

$$\sum_{|z_j|\le r} \frac{1}{|z_j|^\mu} = \mu \int_0^r \frac{n(t)}{t^{\mu+1}} dt + \frac{n(r)}{r^\mu}.$$

Proof. We proceed as in the proof of Lemma 3.2 and find that

$$\sum_{|z_j|\le r} \frac{1}{|z_j|^\mu} = \sum_{|z_j|\le r} \left(\mu \int_{|z_j|}^r \frac{dt}{t^{\mu+1}} + \frac{1}{r^\mu} \right) = \mu \int_0^r \frac{n(t)}{t^{\mu+1}} dt + \frac{n(r)}{r^\mu}. \qquad \square$$

Remark. A proof using the Riemann-Stieltjes integral is also instructive:

$$\begin{aligned} \sum_{|z_j|\le r} \frac{1}{|z_j|^\mu} &= \int_\varepsilon^r \frac{1}{t^\mu} dn(t) \\ &= \frac{1}{t^\mu} n(t) \Big|_\varepsilon^r - \int_\varepsilon^r n(t) d\left(\frac{1}{t^\mu}\right) \\ &= \frac{n(r)}{r^\mu} + \mu \int_\varepsilon^r \frac{n(t)}{t^{\mu+1}} dt. \end{aligned}$$

Here $0 < \varepsilon < \min_j |z_j|$.

Lemma 9.2. *Let (z_j) and $n(r)$ be as in Theorem 9.1, and let $\mu > 0$. Then*

$$\sum_{j=1}^\infty \frac{1}{|z_j|^\mu} < \infty$$

if and only if

$$\int_0^\infty \frac{n(t)}{t^{\mu+1}} dt < \infty.$$

In this case

$$\lim_{r\to\infty} \frac{n(r)}{r^\mu} = 0.$$

Proof. Suppose first that the series converges. Lemma 9.1 implies that the integral also converges and the limit $\lim_{r\to\infty} n(r)/r^\mu = 0$ exists. But since $\int_1^\infty dt/t = \infty$, the existence of the limit and the comparison test imply that the limit is actually equal to 0.

Suppose now that the integral converges. Using the comparison test as before we see that $\liminf_{r\to\infty} n(r)/r^\mu = 0$. Lemma 9.1 now yields that the series converges, and as before this yields that in fact $\lim_{r\to\infty} n(r)/r^\mu = 0$. □

Proof of Theorem 9.1. Suppose that $\sigma < \infty$ and choose μ with $\sigma < \mu < \infty$. Then

$$\sum_{j=1}^\infty \frac{1}{|z_j|^\mu} < \infty.$$

Lemma 9.2 implies that

$$\lim_{r\to\infty} \frac{n(r)}{r^\mu} = 0.$$

We thus have $n(r) \le r^\mu$ and hence $\log n(r) \le \mu \log r$ for large r and hence

$$\limsup_{r\to\infty} \frac{\log n(r)}{\log r} \le \mu.$$

As $\mu \in (\sigma, \infty)$ was arbitrary, this yields that

$$\limsup_{r\to\infty} \frac{\log n(r)}{\log r} \le \sigma$$

Trivially this holds if $\sigma = \infty$.

Suppose now that

$$\tau := \limsup_{r\to\infty} \frac{\log n(r)}{\log r} < \infty.$$

We want to show that $\sigma \le \tau$. In order to do so, choose ν, μ with $\tau < \nu < \mu < \infty$. For large r we then have $\log n(r) \le \nu \log r$ and hence $n(r) \le r^\nu$. This implies that

$$\int_0^\infty \frac{n(t)}{t^{\mu+1}} < \infty.$$

Lemma 9.2 now yields that

$$\sum_{j=1}^\infty \frac{1}{|z_j|^\mu} < \infty.$$

Since μ can be chosen arbitrarily close to τ we conclude that $\tau \leq \sigma$. For $\sigma = \infty$ this is trivial. This proves the first equation claimed in Theorem 9.1.

The second equation follows easily from Lemma 3.3 which yields that

$$N(r) \leq n(r) \log r + \mathcal{O}(1)$$

and

$$n(r) \leq \frac{1}{\log K} N(Kr)$$

if $K > 1$. We omit the details. □

Theorem 9.2. *Let f be meromorphic, $a \in \widehat{\mathbb{C}}$, and let (z_j) be the sequence of a-points of f. Then $\sigma((z_j)) \leq \rho(f)$.*

Proof. By Theorem 9.1 we have

$$\sigma((z_j)) = \limsup_{r\to\infty} \frac{\log N(r,a)}{\log r}.$$

By the first fundamental theorem, we have $N(r,a) \leq T(r,f) + \mathcal{O}(1)$. The conclusion follows. □

Remark. We will see later that we have equality in Theorem 9.2 except for at most two values of a. This is Borel's theorem, and such values are called *Borel exceptional values.*

Theorem 9.3. *Let (z_j) be a sequence in $\mathbb{C}\setminus\{0\}$, with $\lim_{j\to\infty} |z_j| = \infty$ and with finite exponent of convergence. Let q be the genus of (z_j) and let*

$$P(z) = \prod_{j=1}^{\infty} E\left(\frac{z}{z_j}, q\right).$$

Then, with $n(r)$ as in Theorem 9.1,

$$\log M(r,P) \leq L(q)\left(r^q \int_0^r \frac{n(t)}{t^{q+1}}dt + r^{q+1}\int_r^\infty \frac{n(t)}{t^{q+2}}dt\right),$$

where

$$L(q) = \begin{cases} 1 & \text{if} \quad q = 0, \\ (q+1)(2+\log q) & \text{if} \quad q \geq 1. \end{cases}$$

Remark. The product for P converges by Theorem 8.3 (and the definition of q). Such a product P is also called *canonical product.*

We will first prove the following lemma.

Lemma 9.3. *Let $q \in \mathbb{N}_0$ and $u \in \mathbb{C}$. Then*

$$\log|E(u,p)| \leq |u|^{q+1} \quad \text{for} \quad |u| \leq 1,\ u \neq 1$$

and, if $q \geq 1$,

$$\log|E(u,p)| \leq M(q)|u|^q \quad \text{for} \quad |u| \geq 1,\ u \neq 1$$

with

$$M(q) = 2 + \log q.$$

Proof. Lemma 8.1 yields if $|u| \le 1$, then

$$\log|E(u,p)| \le \log(1+|E(u,p)-1|) \le |E(u,p)-1| \le |u|^{q+1}.$$

This is the first inequality claimed. To prove the second one, let $q \ge 1$ and $|u| \ge 1$. Then

$$\log|E(u,p)| \le \log|1-u| + \sum_{j=1}^{q} \frac{1}{j}|u|^j \le |u| + \sum_{j=1}^{q} \frac{1}{j}|u|^q \le |u|^q \left(1 + \sum_{j=1}^{q} \frac{1}{j}\right).$$

Since

$$\sum_{j=2}^{q} \frac{1}{j} \le \int_1^q \frac{dt}{t} = \log q$$

the conclusion follows. □

Proof of Theorem 9.3. We have, for z with $|z| = r$,

$$\begin{aligned} \log|P(z)| &= \sum_{j=1}^{\infty} \log\left|E\left(\frac{z}{z_j}, q\right)\right| \\ &= \sum_{|z_j|\le r} \log\left|E\left(\frac{z}{z_j}, q\right)\right| + \sum_{|z_j|>r} \log\left|E\left(\frac{z}{z_j}, q\right)\right| \\ &=: \Sigma_1 + \Sigma_2. \end{aligned}$$

We first estimate Σ_2. By Lemma 9.3 we have

$$\Sigma_2 \le \sum_{|z_j|>r} \left(\frac{r}{|z_j|}\right)^{q+1} = r^{q+1} \sum_{|z_j|>r} \frac{1}{|z_j|^{q+1}}.$$

Lemma 9.1 says that if $R > 0$, then

$$\sum_{r<|z_j|\le R} \frac{1}{|z_j|^{q+1}} = (q+1)\int_r^R \frac{n(t)}{t^{q+2}}dt + \frac{n(R)}{R^{q+1}} - \frac{n(r)}{r^{q+1}}.$$

Since (z_j) has genus q, Lemma 9.2 yields that the integral

$$\int_r^{\infty} \frac{n(t)}{t^{q+2}}dt$$

converges and

$$\lim_{R\to\infty} \frac{n(R)}{R^{q+1}} = 0.$$

It follows that

$$\sum_{|z_j|>r} \frac{1}{|z_j|^{q+1}} = (q+1)\int_r^{\infty} \frac{n(t)}{t^{q+2}}dt - \frac{n(r)}{r^{q+1}}$$

and thus

$$\Sigma_2 \leq (q+1)r^{q+1}\int_r^\infty \frac{n(t)}{t^{q+2}}dt - n(r).$$

If $q = 0$, we have

$$\begin{aligned}
\Sigma_1 &= \sum_{|z_j|\leq r} \log\left|1 - \frac{z}{z_j}\right| \\
&\leq \sum_{|z_j|\leq r} \log\left(1 + \frac{r}{|z_j|}\right) \\
&\leq \sum_{|z_j|\leq r} \log\left(2\frac{r}{|z_j|}\right) \\
&= \sum_{|z_j|\leq r} \log 2 + \sum_{|z_j|\leq r} \log \frac{r}{|z_j|} \\
&= n(r)\log 2 + \int_0^r \frac{n(t)}{t}dt \\
&\leq n(r) + \int_0^r \frac{n(t)}{t}dt
\end{aligned}$$

by Lemma 3.2. Combing the estimates for Σ_1 and Σ_2 gives the conclusion in this case.

If $q \geq 1$, then, by Lemmas 9.3 and 9.1,

$$\begin{aligned}
\Sigma_1 &\leq M(q) \sum_{|z_j|\leq r} \left(\frac{r}{|z_j|}\right)^q \\
&= M(q)r^q \sum_{|z_j|\leq r} \frac{1}{|z_j|^q} \\
&= M(q)r^q \left(q\int_0^r \frac{n(t)}{t^{q+1}}dt + \frac{n(r)}{r^q}\right) \\
&= M(q)q\,r^q \int_0^r \frac{n(t)}{t^{q+1}}dt + M(q)n(r).
\end{aligned}$$

Since $M(q) \geq 1$, the estimate for Σ_2 also yields

$$\Sigma_2 \leq M(q)(q+1)r^{q+1}\int_0^r \frac{n(t)}{t^{q+2}}dt - M(q)n(r).$$

Adding this to the estimate for Σ_1 yields the conclusion, since we may replace $M(q)q$ also by $M(q)(q+1)$ before the first integral. □

Remark. The constants $L(q)$ and $M(q)$ are not sharp.

Theorem 9.4. *Let (z_j), q and P be as in Theorem 9.3; that is, P is the Weierstraß product formed with the z_j, of genus q. Then $\rho(P) = \sigma((z_j))$.*

Proof. By Theorem 9.2 we have $\sigma := \sigma((z_j)) \le \rho(P)$. So we only have to prove the opposite inequality. As remarked after Definition 9.1, we have $q \le \sigma \le q+1$.

We distinguish two cases:

Case 1: $q \le \sigma < q+1$. Choose $\varepsilon > 0$ with $\sigma + \varepsilon < q+1$. Then there exists $K > 0$ such that $n(r) \le K r^{\sigma+\varepsilon}$ for $r > 0$, by Theorem 9.1. Theorem 9.3 implies that

$$\begin{aligned}\log M(r,P) &\le L(q)\cdot K\left(r^q\int_0^r t^{\sigma+\varepsilon-q-1}dt + r^{q+1}\int_r^\infty t^{\sigma+\varepsilon-q-2}dt\right)\\ &= L(q)\cdot K\left(r^q\frac{r^{\sigma+\varepsilon-q}}{\sigma+\varepsilon-q} + r^{q+1}\frac{r^{\sigma+\varepsilon-q-1}}{q+1-\sigma-\varepsilon}\right)\\ &= L(q)\cdot K\left(\frac{1}{\sigma+\varepsilon-q}+\frac{1}{q+1-\sigma-\varepsilon}\right) r^{\sigma+\varepsilon}.\end{aligned}$$

It follows that $\rho(P) \le \sigma+\varepsilon$, and this yields $\rho(P) \le \sigma$ since ε can be chosen arbitrarily small.

Case 2: $\sigma = q+1$. Again we have $n(r) \le K r^{\sigma+\varepsilon}$ with a constant K, for any given $\varepsilon > 0$. By Lemma 9.1, the integral

$$\int_0^\infty \frac{n(t)}{t^{q+2}}dt$$

converges. It follows from Theorem 9.3 that

$$\begin{aligned}\log M(r,P) &\le L(q)\left(Kr^q\int_0^r t^{\sigma+\varepsilon-q-1}dt + r^{q+1}\int_r^\infty \frac{n(t)}{t^{q+2}}dt\right)\\ &= L(q)\cdot K\frac{1}{\sigma+\varepsilon-q}r^{\sigma+\varepsilon} + o\big(r^{q+1}\big).\end{aligned}$$

and hence again that $\rho(P) \le \sigma$. □

Theorem 9.5 (Hadamard factorization theorem)**.** *Let f be meromorphic with $\rho(f) < \infty$. Let P_0 and P_∞ be the canonical products formed with the zeros and poles of f in $\mathbb{C}\setminus\{0\}$, respectively. Let $c_m z^m$ with $c_m \ne 0$ be the first non-vanishing term in the Laurent series of f near 0. Then there exists a polynomial Q with* $\deg Q \le \rho(f)$ *such that*

$$f(z) = z^m e^{Q(z)}\frac{P_0(z)}{P_\infty(z)}.$$

Remark. 1. If $f(0)=0$, then m is the order of this zero. If $f(0)=\infty$, then $-m$ is the order of this pole. Otherwise $m=0$.

2. The main difference to the Weierstraß factorization theorem is the additional hypothesis that f has finite order. This implies that Q is a polynomial (and not just entire).

Proof of Theorem 9.5. The function h defined by

$$h(z) = f(z)\frac{P_\infty(z)}{P_0(z)}z^{-m}$$

is entire and has no zeros. Then it is of the form

$$h(z) = e^{Q(z)}$$

with an entire function Q. Moreover, by Theorem 7.4,

$$\rho(h) \le \max\left\{\rho(f), \rho(P_\infty), \rho(P_0), \rho\big(z^{-m}\big)\right\}.$$

Now $\rho(z^{-m}) = 0$ and denoting by (z_j) and (p_j) the sequences of zeros and poles in $\mathbb{C}\setminus\{0\}$ we have

$$\rho(P_0) = \sigma((z_j)) \le \rho(f)$$

and

$$\rho(P_\infty) = \sigma((p_j)) \le \rho(f).$$

Hence

$$\rho(h) \le \rho(f).$$

It follows that for $\varepsilon > 0$ we have

$$\log M(r,h) \le r^{\rho(f)+\varepsilon}$$

for large r. Since

$$\log M(r,f) = \log \max_{|z|=r}\left|e^{Q(z)}\right| = \log \max_{|z|=r} e^{\operatorname{Re} Q(z)} = A(r,Q)$$

we thus have

$$A(r,Q) \le r^{\rho(f)+\varepsilon}$$

for large r. Theorem 6.10 yields that Q is a polynomial and $\deg Q \le \rho(f)+\varepsilon$. The conclusion follows since ε can be taken arbitrarily small. $\square$

Example 9.1. Let $f(z) = \sin \pi z$. The zeros are $0, \pm 1, \pm 2, \ldots$. Since

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{1}{|k|^\mu} < \infty \quad \text{for} \quad \mu > 1$$

while

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{1}{|k|^\mu} = \infty \quad \text{for} \quad \mu \le 1,$$

the exponent of convergence and the genus of the sequence of zeros are both equal to 1. Moreover, $\rho(f) = 1$ and f has a simple zero at 0. By Hadamard's factorization theorem there exits a polynomial Q of degree at most 1 such that

$$f(z) = ze^{Q(z)} \prod_{j\in\mathbb{Z}\setminus\{0\}} E\left(\frac{z}{z_j}, 1\right).$$

Writing $Q(z) = az + b$ we have

$$\begin{aligned}\sin \pi z &= z e^{az+b} \prod_{j \in \mathbb{Z}\setminus\{0\}} \left(1 - \frac{z}{j}\right) e^{-z/j} \\ &= z e^{az+b} \prod_{j=1}^{\infty} \left(1 - \frac{z}{j}\right) e^{-z/j} \left(1 + \frac{z}{j}\right) e^{z/j} \\ &= z e^{az+b} \prod_{j=1}^{\infty} \left(1 - \frac{z^2}{j^2}\right).\end{aligned}$$

In order to determine a and b we write this as

$$e^{az+b} = \frac{\sin \pi z}{z} \cdot \frac{1}{\prod_{j=1}^{\infty} (1 - z^2/j^2)}.$$

We conclude that

$$e^{az+b} = e^{a(-z)+b} = e^{-az+b}$$

for all $z \in \mathbb{C}$ and thus $a = 0$. Hence

$$e^b = \frac{\sin \pi z}{z} \frac{1}{\prod_{j=1}^{\infty} (1 - z^2/j^2)}$$

for all $z \in \mathbb{C}$, where "for all $z \in \mathbb{C}$" is understood to mean that if the expression on the right hand side has a removable singularity, then it has to be replaced by the appropriate value. It follows that

$$e^b = \lim_{z \to 0} \frac{\sin \pi z}{z} \cdot \frac{1}{\prod_{j=1}^{\infty}(1 - z^2/j^2)} = \lim_{z \to 0} \frac{\sin \pi z}{z} = \pi.$$

Altogether we find that

$$\sin \pi z = \pi z \prod_{j=1}^{\infty} \left(1 - \frac{z^2}{j^2}\right).$$

E.g. for $z = \frac{1}{2}$ we obtain

$$\begin{aligned}\pi &= \frac{\sin \frac{1}{2}\pi}{\frac{1}{2} \prod_{j=1}^{\infty} \left(1 - \frac{1}{4j^2}\right)} \\ &= \frac{2}{\prod_{j=1}^{\infty} \frac{4j^2 - 1}{4j^2}} \\ &= 2 \cdot \prod_{j=1}^{\infty} \frac{4j^2}{4j^2 - 1} \\ &= 2 \cdot \prod_{j=1}^{\infty} \frac{2j}{2j-1} \cdot \frac{2j}{2j+1} \\ &= 2 \cdot \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \ldots.\end{aligned}$$

This formula for π is known as Wallis' product.

Example 9.2. Let f be meromorphic without zeros and poles at $0, -1, -2, \dots$ (and no other poles). The exponent of convergence and genus of the poles are 1. Thus $\rho(f) \geq 1$. Assuming in addition that $\rho(f) = 1$, we see that the function f with the above properties are precisely those which have the form

$$\begin{aligned} f(z) &= z^{-1} e^{az+b} \frac{1}{\prod_{j=1}^{\infty}(1+\frac{z}{j})e^{-z/j}} \\ &= \frac{e^{az+b}}{z\prod_{j=1}^{\infty}(1+\frac{z}{j})e^{-z/j}} \\ &= \frac{e^{az+b}}{z} \cdot \prod_{j=1}^{\infty} \frac{je^{z/j}}{j+z} \end{aligned}$$

with $a, b \in \mathbb{C}$.

The functions $zf(z)$ and $f(z+1)$ have the same poles (in $-1, -2, -3, \dots$) and no zeros. Their quotient $zf(z)/f(z+1)$ is thus entire and without zeros. Moreover, it has order at most 1 and is thus of the form e^{cz+d} with $c, d \in \mathbb{C}$. More precisely, we have

$$\begin{aligned} \frac{zf(z)}{f(z+1)} &= \frac{e^{az+d}(z+1)}{e^{a(z+1)+b}} \prod_{j=1}^{\infty} \frac{j}{j+z} \frac{j+z+1}{j} \frac{e^{z/j}}{e^{(z+1)/j}} \\ &= e^{-a}(z+1) \lim_{n\to\infty} \prod_{j=1}^{n} \frac{j+z+1}{j+z} e^{-1/j} \\ &= e^{-a} \lim_{n\to\infty} (z+n+1) \exp\left(-\sum_{j=1}^{n}\frac{1}{j}\right) \\ &= e^{-a} \lim_{n\to\infty} \frac{z+n+1}{n} \exp\left(\log n - \sum_{j=1}^{n}\frac{1}{j}\right) \\ &= e^{-a} \lim_{n\to\infty} \exp\left(\log n - \sum_{j=1}^{n}\frac{1}{j}\right) \\ &= e^{-a}e^{\gamma} \end{aligned}$$

where

$$\gamma = \lim_{n\to\infty}\left(\log n - \sum_{j=1}^{n}\frac{1}{j}\right).$$

The limit $\gamma = 0.57721\dots$ is called the Euler-Mascheroni constant (or Euler's constant).

For a detailed proof that the limit exists one may write

$$\sum_{j=1}^{n}\frac{1}{j} - \log n = \int_1^n \left(\frac{1}{[x]} - \frac{1}{x}\right) dx + \frac{1}{n}$$

and note that, since

$$0 \le \frac{1}{[x]} - \frac{1}{x} = \frac{x-[x]}{x[x]} \le \frac{1}{x[x]} \le \frac{2}{x^2} \quad \text{for} \quad x \ge 1$$

the integral

$$\int_1^\infty \left(\frac{1}{[x]} - \frac{1}{x} \right) dx$$

converges.

Choosing $a = -\gamma$ in the definition of f we thus have $f(z+1) = zf(z)$. Choosing $b = 0$ we have

$$f(1) = \lim_{z\to 0} zf(z) = \lim_{z\to 0} \frac{e^{-\gamma z}}{\prod_{j=1}^\infty (1+z/j)e^{-z/j}} = 1.$$

We conclude that $f(1) = 1$, $f(2) = 1$, $f(3) = 2 \cdot 1 = 2$, $f(4) = 3 \cdot f(3) = 3 \cdot 2$ and in general

$$f(n) = (n-1)! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1)$$

for $n \in \mathbb{N}$. The function f obtained is called the Gamma function and denoted by Γ, that is

$$\Gamma(z) = \frac{e^{-\gamma z}}{z \prod_{j=1}^\infty \left(1 + \frac{z}{j}\right) e^{-z/j}} = \frac{e^{-\gamma z}}{z} \prod_{j=1}^\infty \frac{je^{z/j}}{j+z}.$$

As noted above, we have $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$ and $\Gamma(z+1) = z\Gamma(z)$ for $z \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$.

Remark. For $\operatorname{Re} z > 0$ the Gamma function is often defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt.$$

It can be shown that this agrees with our definition. The functional equation $\Gamma(z+1) = z\Gamma(z)$ follows from this via integration by parts.

10 Preparations for the second fundamental theorem

The first fundamental theorem says that

$$T(r,f) = N(r,f) + m(r,f) = N\left(r, \frac{1}{f-a}\right) + m\left(r, \frac{1}{f-a}\right) + \mathcal{O}(1)$$

as $r \to \infty$, for all $a \in \mathbb{C}$. So if $T(r,f)$ is large, then one of the terms $N(r, 1/(f-a))$ or $m(r, 1/(f-a))$ must be large, meaning that f has many a-points or f is close to a on some part of the circle $\partial D(0,r)$.

The second fundamental theorem will say that for "most" values of a the first alternative will hold. More precisely, we will see that if $a_1, \dots, a_q \in \mathbb{C}$, then

$$\sum_{j=1}^{q} m\left(r, \frac{1}{f - a_j}\right) + m(r, f) \leq 2T(r, f) + \dots$$

where the dots indicate a "small error term". Together with the first fundamental theorem this yields

$$(q-1)T(r, f) \leq \sum_{j=1}^{q} N\left(r, \frac{1}{f - a_j}\right) + N(r, f) + \dots.$$

In particular, this leads to a contradiction if $f(z) \neq a_1, a_2, \infty$ for all $z \in \mathbb{C}$ so that Picard's theorem follows. However, the result gives much stronger results than just Picard's theorem.

We will first give an upper bound for the difference of the left and right side of the above inequalities. Later we will then show that the upper bound obtained is indeed small in a suitable sense.

Theorem 10.1. *Let f be meromorphic (and non-constant) and let $a_1, \dots, a_q \in \mathbb{C}$ be distinct. Then*

$$\sum_{j=1}^{q} m\left(r, \frac{1}{f - a_j}\right) + m(r, f) \leq 2T(r, f) - N_1(r) + S(r)$$

where

$$N_1(r) = N\left(r, \frac{1}{f'}\right) + 2N(r, f) - N(r, f') \geq 0$$

for $r \geq 1$ and

$$S(r) \leq m\left(r, \frac{f'}{f}\right) + \sum_{j=1}^{q} m\left(r, \frac{f'}{f - a_j}\right) + \mathcal{O}(1)$$

as $r \to \infty$.

Remark. A pole of order k is counted $2k$ times in $2N(r, f)$ and $k + 1$ times in $N(r, f')$. Altogether it is then counted $2k - (k + 1)$ times; that is, $k - 1$ times. It follows that $N_1(r) \geq 0$ for $r \geq 1$.

An a-points of multiplicity $k \geq 2$ is a zero of f' of multiplicity $k - 1$ and thus counted $k - 1$ times in $N(r, 1/f')$. Together with the above considerations for poles we see that $N_1(r)$ is a term that counts the multiple values of f.

Proof of Theorem 10.1. Put

$$P(w) = \prod_{j=1}^{q} (w - a_j).$$

By Theorem 5.3 we have

$$T(r, P \circ f) = q \cdot T(r, f) + \mathcal{O}(1).$$

Since

$$N\left(r, \frac{1}{P \circ f}\right) = \sum_{j=1}^{q} N\left(r, \frac{1}{f - a_j}\right)$$

the first fundamental theorem yields that

$$\begin{aligned} m\left(r, \frac{1}{P \circ f}\right) &= T(r, P \circ f) - N\left(r, \frac{1}{P \circ f}\right) + \mathcal{O}(1) \\ &= q \cdot T(r, f) - \sum_{j=1}^{q} N\left(r, \frac{1}{f - a_j}\right) + \mathcal{O}(1) \\ &= \sum_{j=1}^{q} \left(T(r, f) - N\left(r, \frac{1}{f - a_j}\right)\right) + \mathcal{O}(1) \\ &= \sum_{j=1}^{q} m\left(r, \frac{1}{f - a_j}\right) + \mathcal{O}(1). \end{aligned}$$

A partial fraction decomposition yields

$$\frac{1}{P(w)} = \sum_{j=1}^{q} \frac{c_j}{w - a_j}$$

with constants $c_j \in \mathbb{C} \setminus \{0\}$. (In fact, we have $c_j = 1/P'(a_j)$ for $1 \le j \le q$.) It follows, using Lemma 4.1, that

$$\begin{aligned} m\left(r, \frac{1}{P \circ f}\right) &= m\left(r, \frac{f'}{P \circ f} \cdot \frac{1}{f'}\right) + \mathcal{O}(1) \\ &\le m\left(r, \frac{f'}{P \circ f}\right) + m\left(r, \frac{1}{f'}\right) + \mathcal{O}(1) \\ &= m\left(r, \sum_{j=1}^{q} c_j \frac{f'}{f - a_j}\right) + m\left(r, \frac{1}{f'}\right) + \mathcal{O}(1) \\ &\le \sum_{j=1}^{q} m\left(r, \frac{f'}{f - a_j}\right) + \sum_{j=1}^{q} \log^+ |c_j| + \log q + m\left(r, \frac{1}{f'}\right) + \mathcal{O}(1). \end{aligned}$$

Combining the above estimates we have

$$\begin{aligned} \sum_{j=1}^{q} m\left(r, \frac{1}{f - a_j}\right) &= m\left(r, \frac{1}{P \circ f}\right) + \mathcal{O}(1) \\ &\le m\left(r, \frac{1}{f'}\right) + \sum_{j=1}^{q} m\left(r, \frac{f'}{f - a_j}\right) + \mathcal{O}(1). \end{aligned}$$

The first fundamental theorem yields that

$$\begin{aligned} m\left(r, \frac{1}{f'}\right) &= T(r, f') - N\left(r, \frac{1}{f'}\right) + \mathcal{O}(1) \\ &= N(r, f') + m(r, f') - N\left(r, \frac{1}{f'}\right) + \mathcal{O}(1) \\ &= 2N(r, f) - N_1(r) + m(r, f') + \mathcal{O}(1) \\ &= 2N(r, f) - N_1(r) + m\left(r, \frac{f'}{f} f\right) + \mathcal{O}(1) \\ &\leq 2N(r, f) - N_1(r) + m\left(r, \frac{f'}{f}\right) + m(r, f) + \mathcal{O}(1) \end{aligned}$$

Together with the previous inequality we thus have

$$\begin{aligned} \sum_{j=1}^{q} m\left(r, \frac{1}{f - a_j}\right) \leq 2N(r, f) + m(r, f) - N_1(r) + m\left(r, \frac{f'}{f}\right) \\ + \sum_{j=1}^{q} m\left(r, \frac{f'}{f - a_j}\right) + \mathcal{O}(1). \end{aligned}$$

Adding $m(r, f)$ on both sides and noting that $T(r, f) = m(r, f) + N(r, f)$ by definition yields the conclusion. □

11 The lemma on the logarithmic derivative

Theorem 10.1 shows that the second fundamental theorem requires an estimate of $m(r, f'/f)$. Since $f'/f = (\log f)'$, the term f'/f is called the *logarithmic derivative*. In order to obtain the desired estimate, we begin with the following representation of the logarithmic derivative.

Theorem 11.1. *Let f be meromorphic in $\overline{D}(0, r)$ and let $a_1, \ldots, a_m$ be the zeros and $b_1, \ldots, b_n$ the poles of f in $D(0, r)$. Let $z \in D(0, r)$ with $f(z) \neq 0, \infty$. Then*

$$\begin{aligned} \frac{f'(z)}{f(z)} =& \frac{1}{2\pi} \int_0^{2\pi} \log \left| f\left(re^{i\theta}\right) \right| \frac{2re^{i\theta}}{(re^{i\theta} - z)^2} d\theta \\ &+ \sum_{j=1}^{m} \left(\frac{1}{z - a_j} + \frac{\overline{a_j}}{r^2 - \overline{a_j} z} \right) - \sum_{j=1}^{n} \left(\frac{1}{z - b_j} + \frac{\overline{b_j}}{r^2 - \overline{b_j} z} \right). \end{aligned}$$

Proof. Theorem 6.8 says that if g is holomorphic in $\overline{D}(0, r)$, then

$$g(z) = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} g\left(re^{i\theta}\right) \frac{re^{i\theta} + z}{re^{i\theta} - z} d\theta + i \operatorname{Im} g(0).$$

Differentiating this with respect to z we obtain

$$g'(z) = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} g\left(re^{i\theta}\right) \frac{2re^{i\theta}}{(re^{i\theta} - z)^2} d\theta.$$

This yields the conclusion if f has no zeros and poles, since then f may be written in the form $f = e^g$ with a holomorphic function f. Then $f'/f = g$ and $\log|f| = \operatorname{Re} g$. In the general case we assume again for simplicity that f has no zeros and poles on $\partial D(0, r)$ and, as in the proof of Theorem 3.2 and 6.2, consider

$$h(z) = f(z)\frac{\prod_{j=1}^{n} \varphi_{b_j}(z)}{\prod_{j=1}^{m} \varphi_{a_j}(z)}$$

with

$$\varphi_a(z) = \frac{r(z-a)}{r^2 - \overline{a}z}.$$

Then h has no zeros and poles in $\overline{D}(0, r)$ and $|h(re^{i\theta})| = |f(re^{i\theta})|$ for $\theta \in \mathbb{R}$. Using the result already proved we conclude that

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(re^{i\theta}\right)\right| \frac{2r}{(re^{i\theta}-z)^2}d\theta &= \frac{h'(z)}{h(z)} \\ &= \frac{f'(z)}{f(z)} + \sum_{j=1}^{n}\frac{\varphi'_{b_j}(z)}{\varphi_{b_j}(z)} - \sum_{j=1}^{m}\frac{\varphi'_{a_j}(z)}{\varphi_{a_j}(z)} \\ &= \frac{f'(z)}{f(z)} + \sum_{j=1}^{n}\left(\frac{1}{z-b_j} + \frac{\overline{b_j}}{r^2 - \overline{b_j}z}\right) \\ &\quad - \sum_{j=1}^{m}\left(\frac{1}{z-a_j} + \frac{\overline{a_j}}{r^2 - \overline{a_j}z}\right)\end{aligned}$$

as claimed. □

Remark. In the last equation we have used that if $F = f_1 \cdot f_2$ and $G = f_1/f_2$, then

$$\frac{F'}{F} = \frac{f_1'}{f_1} + \frac{f_2'}{f_2} \quad \text{and} \quad \frac{G'}{G} = \frac{f_1'}{f_1} - \frac{f_2'}{f_2}.$$

These rules can be proved (and remembered) by noting that locally we may write $\log F = \log f_1 + \log f_2$ with suitably chosen branches of the logarithm so that

$$\frac{F'}{F} = (\log F)' = (\log f_1)' + (\log f_2)' = \frac{f_1'}{f_1} + \frac{f_2'}{f_2},$$

with an analogous reasoning for G'/G. Of course, the above rules can also be checked with the product and quotient rules.

Theorem 11.2 (Jensen's inequality)**.** *Let $I \subset \mathbb{R}$ be an interval and $f\colon I \to \mathbb{R}$ convex. Let $\lambda_1, \ldots, \lambda_n \in (0, 1)$ with $\sum_{j=1}^{n} \lambda_j = 1$ and $x_1, \ldots, x_n \in I$. Then*

$$f\left(\sum_{j=1}^{n} \lambda_j x_j\right) \leq \sum_{j=1}^{n} \lambda_j f(x_j).$$

Proof. The conclusion is trivial if all x_j are equal. Suppose that this is not the case. Then

$$\xi := \sum_{j=1}^{n} \lambda_j x_j$$

is in the interior of I. Since f is convex, there exists an affine function g such that $g(\xi) = f(\xi)$ and $g(t) \leq f(t)$ for all $t \in I$. Here by affine we mean that g has the form $g(t) = at + b$ for certain $a, b \in \mathbb{R}$. It follows that

$$\begin{aligned} f(\xi) = g(\xi) = a\xi + b &= a\left(\sum_{j=1}^{n} \lambda_j x_j\right) + b \\ &= a\left(\sum_{j=1}^{n} \lambda_j x_j\right) + b\sum_{j=1}^{n} \lambda_j \\ &= \sum_{j=1}^{n} \lambda_j (a x_j + b) \\ &= \sum_{j=1}^{n} \lambda_j g(x_j) \\ &\leq \sum_{j=1}^{n} \lambda_j f(x_j). \end{aligned}$$

□

Theorem 11.3 (Jensen's inequality, version for integrals)**.** *Let $I \subset \mathbb{R}$ be an interval, $f\colon I \to \mathbb{R}$ convex and $g\colon [a,b] \to I$ (Riemann) integrable. Then*

$$f\left(\frac{1}{b-a}\int_a^b g(t)dt\right) \leq \frac{1}{b-a}\int_a^b f(g(t))dt.$$

Remark. Since convex functions are continuous, it follows from the hypothesis of the theorem that $f \circ g$ is also integrable.

Proof of Theorem 11.3. Let $a \leq t_0 < t_1 < \ldots < t_n = b$ be a partition of the interval $[a,b]$ and let $\tau_j \in [t_{j-1}, t_j]$ for $j = 1, \ldots, n$. With

$$\lambda_j = \frac{t_j - t_{j-1}}{b-a} \quad \text{and} \quad x_j = g(\tau_j)$$

we deduce from Theorem 11.2 that

$$f\left(\frac{1}{b-a}\sum_{j=1}^{n}(t_j - t_{j-1})g(\tau_j)\right) \leq \frac{1}{b-a}\sum_{j=1}^{n}(t_j - t_{j-1})f(g(\tau_j)).$$

The conclusion follows by taking the limit through a sequence of partitions with maximal interval length tending to 0, using also that f is continuous. □

Remark. The result can be extended to Lebesgue integrable function, e.g. by approximation by continuous functions. Using $f = \exp$ and $g = \log h$ one obtains the following result.

Corollary. Let h and $\log h$ be integrable over $[a,b]$. Then

$$\frac{1}{b-a}\int_a^b \log h(t)dt \le \log\left(\frac{1}{b-a}\int_a^b h(t)dt\right).$$

The proof of the following simple lemma is omitted.

Lemma 11.1. *Let* $0<\alpha<1$ *and* $x_1,\dots,x_n \ge 0$. *Then*

$$\left(\sum_{j=1}^n x_j\right)^\alpha \le \sum_{j=1}^n x_j^\alpha.$$

Theorem 11.4. *Let* f *be meromorphic with* $f(0)=1$ *and let* $0<r<R$ *and* $0<\alpha<1$. *Then*

$$m\left(r,\frac{f'}{f}\right) \le \frac{1}{\alpha}\log\left(1+\frac{24}{1-\alpha}\left(\frac{R}{R-r}\right)^{1+\alpha}\frac{T(R,f)}{r^\alpha}\right).$$

Proof. Let $r<\rho<R$ and $z\in D(0,\rho)$. Theorem 11.1 yields that if $f(z)\neq 0,\infty$, then

$$\begin{aligned}\left|\frac{f'(z)}{f(z)}\right| \le & \frac{1}{2\pi}\int_0^{2\pi}\left|\log\left|f\left(\rho e^{i\theta}\right)\right|\right|\frac{2\rho}{(\rho-|z|)^2}d\theta + \sum_{|a_j|<\rho}\left|\frac{1}{z-a_j}+\frac{\overline{a_j}}{\rho^2-\overline{a_j}z}\right| \\ & + \sum_{|b_j|<\rho}\left|\frac{1}{z-b_j}+\frac{\overline{b_j}}{\rho^2-\overline{b_j}z}\right|.\end{aligned}$$

Here the a_j and b_j are the zeros and poles of f. We combine the sequences of zeros and poles into one sequence (c_j). Noting that

$$|\log|w|| = \log^+|w| + \log^+\frac{1}{|w|}$$

for $w\in\mathbb{C}\setminus\{0\}$ we thus conclude that

$$\left|\frac{f'(z)}{f(z)}\right| \le \frac{2\rho}{(\rho-|z|)^2}\left(m(\rho,f)+m\left(\rho,\frac{1}{f}\right)\right) + \sum_{|c_j|<\rho}\left|\frac{1}{z-c_j}+\frac{\overline{c_j}}{\rho^2-\overline{c_j}z}\right|.$$

For $c\in D(0,\rho)$ we have

$$\left|\frac{1}{z-c}+\frac{\bar{c}}{\rho^2-\bar{c}z}\right| = \left|\frac{1}{z-c}\left(1+\frac{\bar{c}}{\rho}\frac{\rho(z-c)}{\rho^2-\bar{c}z}\right)\right|.$$

Since the map given by

$$z\mapsto \frac{\rho(z-c)}{\rho^2-\bar{c}z}$$

maps $D(0,\rho)$ onto $D(0,1)$ by Lemma 3.1, we deduce that

$$\left|\frac{1}{z-c}+\frac{\overline{c}}{\rho^2-\overline{c}z}\right| \le \frac{1}{|z-c|}\left(1+\frac{|c|}{\rho}\right) \le \frac{2}{|z-c|}.$$

We conclude, noting that $m(\rho,1/f)\le T(r,f)$ since $f(0)=1$, that

$$\left|\frac{f'(z)}{f(z)}\right| \le \frac{4\rho}{(\rho-|z|)^2}T(\rho,f)+2\sum_{|c_j|<\rho}\frac{1}{|z-c_j|}.$$

Together with Lemma 11.1 this yields

$$\begin{aligned}\log^+\left|\frac{f'(z)}{f(z)}\right| &\le \log\left(1+\left|\frac{f'(z)}{f(z)}\right|\right)\\ &= \frac{1}{\alpha}\log\left(\left(1+\left|\frac{f'(z)}{f(z)}\right|\right)^\alpha\right)\\ &\le \frac{1}{\alpha}\log\left(1+\frac{(4\rho)^\alpha}{(\rho-|z|)^{2\alpha}}T(\rho,f)^\alpha+2^\alpha\sum_{|c_j|<\rho}\frac{1}{|z-c_j|^\alpha}\right).\end{aligned}$$

Integrating this, using Jensen's inequality, and also noting that $4^\alpha<4$ and $2^\alpha<2$ we obtain

$$\begin{aligned}m\left(r,\frac{f'}{f}\right) &\le \frac{1}{\alpha}\frac{1}{2\pi}\int_0^{2\pi}\log\left(1+\frac{4\rho^\alpha}{(\rho-r)^{2\alpha}}T(\rho,f)^\alpha+2\sum_{|c_j|<\rho}\frac{1}{|re^{i\theta}-c_j|^\alpha}\right)d\theta\\ &\le \frac{1}{\alpha}\log\left(1+\frac{4\rho^\alpha}{(\rho-r)^{2\alpha}}T(\rho,f)^\alpha+2\sum_{|c_j|<\rho}\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{|re^{i\theta}-c_j|^\alpha}\right).\end{aligned}$$

Write $c_j=|c_j|e^{i\varphi_j}$. Then

$$\begin{aligned}\int_0^{2\pi}\frac{d\theta}{|re^{i\theta}-c_j|^\alpha} &= \int_0^{2\pi}\frac{d\theta}{|re^{i\theta}-|c_j|e^{i\varphi_j}|^\alpha}\\ &= \int_0^{2\pi}\frac{d\theta}{|re^{i\theta-\varphi_j}-|c_j||^\alpha}\\ &= \int_0^{2\pi}\frac{dt}{|re^{it}-|c_j||^\alpha}\\ &\le \int_0^{2\pi}\frac{dt}{r^\alpha|\sin t|^\alpha}\\ &= \frac{4}{r^\alpha}\int_0^{\pi/2}\frac{dt}{|\sin t|^\alpha}.\end{aligned}$$

Since $\sin t \geq 2t/\pi$ for $0 \leq t \leq \pi/2$ we thus have

$$\begin{aligned}
\int_0^{2\pi} \frac{d\theta}{|re^{i\theta} - c_j|^\alpha} &\leq \frac{4}{r^\alpha} \cdot \frac{2^\alpha}{\pi^\alpha} \int_0^{\pi/2} \frac{dt}{t^\alpha} \\
&= \frac{4}{r^\alpha} \cdot \frac{2^\alpha}{\pi^\alpha} \cdot \frac{t^{1-\alpha}}{1-\alpha}\bigg|_{t=1}^{t=\pi/2} \\
&\leq \frac{4}{r^\alpha} \cdot \frac{\pi}{2(1-\alpha)} \\
&= \frac{2\pi}{r^\alpha(1-\alpha)}.
\end{aligned}$$

Inserting this into the above estimate for $m(r, f'/f)$ we obtain

$$m\left(r, \frac{f'}{f}\right) \leq \log\left(1 + \frac{4\rho^\alpha}{(\rho - r)^{2\alpha}} T(\rho, f)^\alpha + \frac{2}{r^\alpha(1-\alpha)}\left(n(\rho, f) + n\left(\rho, 1/f\right)\right)\right).$$

We now choose $\rho = \frac{1}{2}(R + r)$. Then $\rho \leq R$ and $\rho - r = \frac{1}{2}(R - r)$ and thus

$$\frac{4\rho^\alpha}{(\rho - r)^{2\alpha}} T(\rho, f)^\alpha \leq \frac{16R^\alpha}{(R - r)^\alpha} T(R, f).$$

We also have, by Lemma 3.3,

$$N(R, f) \geq \int_\rho^R \frac{n(t, f)}{t} dt \geq n(\rho, f) \log \frac{R}{\rho}.$$

It is easy to see that

$$\log x \geq \frac{x - 1}{x} \quad \text{for} \quad x > 0.$$

This yields that

$$\log \frac{R}{\rho} \geq \frac{R/\rho - 1}{R/\rho} = \frac{R - \rho}{R} = \frac{R - r}{2R}$$

and hence that

$$n(\rho, f) \leq \frac{1}{\log \frac{R}{\rho}} N(R, f) \leq \frac{2R}{R - r} N(R, f) \leq \frac{2R}{R - r} T(R, f)$$

and also

$$n\left(\rho, \frac{1}{f}\right) \leq \frac{2R}{R - r} T(R, f).$$

Altogether we thus have

$$m\left(r, \frac{f'}{f}\right) \leq \frac{1}{\alpha} \log\left(1 + \frac{16R^\alpha}{(R - r)^{2\alpha}} T(R, f) + \frac{8R}{(R - r)r^\alpha(1-\alpha)} T(R, f)\right).$$

The conclusion follows since

$$\begin{aligned}\frac{16R^\alpha}{(R-r)^{2\alpha}} + \frac{8R}{(R-r)r^\alpha(1-\alpha)} &= \frac{1}{r^\alpha(1-\alpha)}\left(\frac{16R^\alpha r^\alpha(1-\alpha)}{(R-r)^{2\alpha}} + \frac{8R}{R-r}\right)\\ &\le \frac{1}{r^\alpha(1-\alpha)}\left(16\left(\frac{R}{R-r}\right)^{2\alpha} + 8\frac{R}{R-r}\right)\\ &\le \frac{1}{r^\alpha(1-\alpha)}\cdot 24\left(\frac{R}{R-r}\right)^{1+\alpha}.\end{aligned}$$

In the last estimate we simply used that $1+\alpha \ge \max\{1, 2\alpha\}$. □

The term $R-r$ in the denominator prevents us from taking $R=r$ in Theorem 11.4. However, the Borel lemma (Lemma 6.1) allows to estimate $m(r, f'/f)$ in terms of $T(r,f)$ outside an exceptional set.

We recall a version of this lemma given in the remarks after Lemma 6.1: Let $x_0, y_0 > 1$, $K > 1$, $\nu\colon [x_0,\infty) \to [e^{y_0},\infty)$ continuous and non-decreasing, and $\varphi\colon [y_0,\infty) \to (0,\infty)$ continuous and non-increasing with

$$\int_{y_0}^\infty \varphi(y)dy < \infty.$$

Then there exists a subset E of $[x_0,\infty)$ of finite measure such that

$$\nu(x+\varphi(\log\nu(x))) \le K\nu(x) \quad \text{for} \quad x \notin E.$$

The choice $\varphi(y) = 1/y^2$ yields that

$$\nu\left(x + \frac{1}{(\log\nu(x))^2}\right) \le K\nu(x)$$

for $x \notin E$.

Theorem 11.5. *Let f be a transcendental meromorphic function and $\varepsilon > 0$. Then there exists a subset E of $[0,\infty)$ of finite measure such that*

$$m\left(r, \frac{f'}{f}\right) \le (1+\varepsilon)\left(\log T(r,f) + \log r\right)$$

for $r \notin E$.

Proof. Without loss of generality we may assume that $f(0)=1$, since otherwise we can replace f by $g(z) = cz^p f(z)$ with a suitable $c \in \mathbb{C}\setminus\{0\}$ and $p \in \mathbb{Z}$. Then

$$m\left(r, \frac{g'}{g}\right) = m\left(r, \frac{f'}{f}\right) + \mathcal{O}(1)$$

and

$$T(r,g) = T(r,f) + \mathcal{O}(\log r) = (1+o(1))T(r,f)$$

by Theorem 6.5. Thus

$$\log T(r,g) = \log T(r,f) + o(1),$$

which implies that f satisfies the conclusion if f does.

To apply the Borel lemma as in the remark above with $\nu(r) = T(r,f)$ and $\varphi(y) = 1/y^2$ we put

$$R = r + \frac{1}{(\log T(r,f))^2}.$$

Then $R \le 2r$ for large r so that Theorem 11.4 yields that if $K > 1$, then

$$\begin{aligned}
\alpha \cdot m\left(r, \frac{f'}{f}\right) &\le \log\left(1 + \frac{24}{1-\alpha} R^{1+\alpha} (\log T(r,f))^{2+2\alpha} \frac{T(R,f)}{r^\alpha}\right) \\
&\le \log\left(1 + \frac{24}{1-\alpha} 2^{1+\alpha} r (\log T(r,f))^{2+2\alpha} T(R,f)\right) \\
&\le \log 2 + \log\left(\frac{24}{1-\alpha} 2^{1+\alpha}\right) + \log r \\
&\qquad + (2+2\alpha)\log\log T(r,f) + \log T(R,f) \\
&\le \log 2 + \log\left(\frac{24}{1-\alpha} 2^{1+\alpha}\right) + \log r \\
&\qquad + (2+2\alpha)\log\log T(r,f) + \log T(r,f) + \log K
\end{aligned}$$

for $r \notin E$, where E has finite measure. Thus

$$\alpha \cdot m\left(r, \frac{f'}{f}\right) \le \left(1 + \frac{\varepsilon}{2}\right) \log T(r,f) + \log r$$

for large $r \notin E$. Taking α close to 1 yields the conclusion. □

Remark. If f is rational, then

$$\frac{f'(z)}{f(z)} = \mathcal{O}\left(\frac{1}{|z|}\right)$$

as $z \to \infty$ and thus

$$m\left(r, \frac{f'}{f}\right) = 0$$

for all large r. Therefore it is no restriction to consider only transcendental functions in Theorem 11.5.

If f has finite order $\rho(f)$, then Theorem 11.5 gives

$$m\left(r, \frac{f'}{f}\right) \le (\rho(f) + 1 + \varepsilon) \log r \quad \text{for} \quad r \notin E.$$

The following result improves the constant on the right hand side and – more importantly – says that in this case no exceptional set E is required.

Theorem 11.6. *Let f be a meromorphic function of finite order. Then*

$$\limsup_{r\to\infty} \frac{m(r, f'/f)}{\log r} \leq \max\{\rho(f) - 1, 0\}.$$

Proof. We may again assume without loss of generality that $f(0) = 1$. We apply Theorem 11.4 with $R = 2r$ and obtain

$$m\left(r, \frac{f'}{f}\right) \leq \frac{1}{\alpha} \log\left(1 + \frac{24}{1-\alpha} 2^{1+\alpha} \frac{T(2r, f)}{r^\alpha}\right).$$

Let $\varepsilon > 0$. Then $T(2r, f) \leq (2r)^{\rho(f)+\varepsilon}$ for large r and thus

$$\begin{aligned} m\left(r, \frac{f'}{f}\right) &\leq \frac{1}{\alpha} \log\left(1 + \frac{24}{1-\alpha} 2^{1+\alpha} 2^{\rho(f)+\varepsilon} r^{\rho(f)+\varepsilon-\alpha}\right) \\ &\leq \frac{1}{\alpha}\left(\log^+\left(\frac{24}{1-\alpha} 2^{1+\alpha+\rho(f)+\varepsilon}\right) + \log^+\left(r^{\rho(f)+\varepsilon-\alpha}\right) + \log 2\right) \\ &\leq \frac{\max\{\rho(f)+\varepsilon-\alpha, 0\}}{\alpha} \log r + \mathcal{O}(1) \end{aligned}$$

as $r \to \infty$. Considering the limits as $\alpha \to 1$ and $\varepsilon \to 0$ we obtain the conclusion. □

12 The second fundamental theorem and the deficiency relation

The following result, called the second fundamental theorem of Nevanlinna theory, is an immediate consequence of Theorem 10.1 and the result of Section 11.

Theorem 12.1. *Let f be meromorphic and let $a_1, \ldots, a_q \in \mathbb{C}$ be distinct. Then*

$$\sum_{j=1}^{q} m\left(r, \frac{1}{f - a_j}\right) + m(r, f) \leq 2T(r, f) - N_1(r) + S(r, f)$$

where

$$N_1(r) := N\left(r, \frac{1}{f'}\right) + 2N(r, f) - N(r, f')$$

and the term $S(r, f)$ satisfies the following:

(i) There exists a subset E of $[0, \infty)$ of finite measure such that

$$S(r, f) = \mathcal{O}(\log T(r, f)) + \mathcal{O}(\log r)$$

as $r \to \infty$, $r \notin E$.

(ii) If f has finite order, then $S(r, f) = \mathcal{O}(\log r)$ as $r \to \infty$.

(iii) If f is rational, then $S(r, f) = \mathcal{O}(1)$.

Remark. For transcendental f we have

$$\lim_{r\to\infty} \frac{T(r,f)}{\log r} = \infty$$

by Theorem 6.5. This implies that

$$S(r,f) = o(T(r,f))$$

as $r \to \infty$, $r \notin E$. The conclusions (ii) and (iii) yield that this holds without the exceptional set E if f has finite order.

Proof of Theorem 12.1. Theorem 10.1 says that the conclusion holds with

$$S(r,f) = m\left(r, \frac{f'}{f}\right) + \sum_{j=1}^{q} m\left(r, \frac{f'}{f-a_j}\right) + \mathcal{O}(1).$$

Noting that $T(r, f-a_j) = T(r,f) + \mathcal{O}(1)$ and $f'/(f-a_j) = (f-a_j)'/(f-a_j)$ the conclusion now follows from the results in Section 11. □

The following result is another version of the second fundamental theorem, using the counting functions instead of the proximity functions. Here we write $N(r,a) = N(r, 1/(f-a))$ for $a \in \mathbb{C}$ and $N(r,\infty) = N(r,f)$.

Theorem 12.2. *Let f be meromorphic and $a_1, \ldots, a_q \in \widehat{\mathbb{C}}$ be distinct. Then*

$$(q-2)T(r,f) \le \sum_{j=1}^{q} N(r,a_j) - N_1(r) + S(r,f),$$

with $S(r,f)$ satisfying the conclusion of Theorem 12.1.

Proof. Suppose first that $a_j \in \mathbb{C}$ for all $j \in \{1, \ldots, q\}$. We add $\sum_{j=1}^{q} N(r,a_j)$ to the left and right hand side of the inequality in Theorem 12.1. Noting that

$$N(r,a_j) + m\left(r, \frac{1}{f-a_j}\right) = T(r,f) + \mathcal{O}(1)$$

by the first fundamental theorem we obtain

$$qT(r,f) + m(r,f) \le \sum_{j=1}^{q} N(r,a_j) + 2T(r,f) - N_1(r) + S(r,f).$$

Subtracting $2T(r,f)$ and noting that $m(r,f) \ge 0$ we obtain the conclusion.

Suppose now that $a_j = \infty$ for some j. Without loss of generality we may assume that $a_q = \infty$. Then the above inequality holds with q replaced by $q-1$. Thus

$$(q-1)T(r,f) + m(r,f) \le \sum_{j=1}^{q-1} N(r,a_j) + 2T(r,f) - N_1(r) + S(r,f).$$

Adding $N(r, a_q) = N(r, \infty) = N(r, f)$ on both sides we obtain

$$qT(r,f) \leq \sum_{j=1}^{q} N(r, a_j) + 2T(r,f) - N_1(r) + S(r,f).$$

and thus the conclusion. □

An alternative method to deal with the case that $a_j = \infty$ for some j is to consider $1/(f-c)$ instead of f for some $c \in \mathbb{C}$ with $c \neq a_j$ for all j.

Let $\overline{n}(r,f)$ denote the number of distinct poles of a meromorphic function f in the closed disk $\overline{D}(0,r)$; that is, multiplicities are not taken into account. In analogy to the definition of $N(r,f)$ we put

$$\overline{N}(r,f) = \int_0^r \frac{\overline{n}(t,f) - \overline{n}(0,f)}{t} dt + \overline{n}(0,f) \log r.$$

As before we also write $\overline{n}(r,\infty)$ and $\overline{N}(r,\infty)$ instead of $n(r,f)$ and $N(r,f)$ and we put

$$\overline{n}(r,a) = \overline{n}\left(r, \frac{1}{f-a}\right) \quad \text{and} \quad \overline{N}(r,a) = \overline{N}\left(r, \frac{1}{f-a}\right).$$

An a-point of multiplicity k is thus counted k times in $N(r,a)$, but only once in $\overline{N}(r,a)$. In $N_1(r)$ it is counted $k-1$ times. This implies that

$$\sum_{j=1}^{q} \left(N(r,a_j) - \overline{N}(r,a_j) \right) \leq N_1(r)$$

for $r \geq 1$. Theorem 12.2 then yields the following result.

Theorem 12.3. *Let f be meromorphic and $a_1, \dots, a_q \in \mathbb{C}$ be distinct. Then*

$$(q-2)T(r,f) \leq \sum_{j=1}^{q} \overline{N}(r,a_j) + S(r,f),$$

with $S(r,f)$ as in Theorem 12.1.

Picard's theorem is a simple consequence of Theorem 12.2 (or Theorem 12.3). The following definitions will be used to state some generalizations.

Definition 12.1. Let f be meromorphic and $a \in \widehat{\mathbb{C}}$. Then

$$\delta(a) = \delta(a,f) = 1 - \limsup_{r\to\infty} \frac{N(r,a)}{T(r,f)}$$

is called the *(Nevanlinna) deficiency* of a, and

$$\varepsilon(a) = \varepsilon(a,f) = \liminf_{r\to\infty} \frac{N(r,a) - \overline{N}(r,a)}{T(r,f)}$$

is called the *ramification index* of a. Moreover, we put

$$\Theta(a) = \Theta(a,f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,a)}{T(r,f)}.$$

Remark. By the first fundamental theorem, we have $0 \leq \delta(a) \leq 1$. If f has no a-points, then $\delta(a) = 1$. In general, the deficiency $\delta(a)$ should be thought of as a measure of how often f takes the value a. A "large" deficiency means that f takes the value relatively "few" times, and vice versa.

A similar interpretation applies to $\Theta(a)$. Analogously, $\varepsilon(a) > 0$ means that f has relatively "many" multiple a-points.

It follows easily from the definition that

$$0 \leq \delta(a) + \varepsilon(a) \leq \Theta(a) \leq 1$$

for all $a \in \widehat{\mathbb{C}}$.

In analogy to the terminology $n(r,a)$ we also put

$$m(r,a) = m\left(r, \frac{1}{f-a}\right)$$

if $a \in \mathbb{C}$ and $m(r,\infty) = m(r,f)$. The first fundamental theorem thus takes the form

$$N(r,a) + m(r,a) = T(r,f) + \mathcal{O}(1)$$

for all $a \in \widehat{\mathbb{C}}$. This implies that

$$\delta(a,f) = \liminf_{r\to\infty} \frac{m(r,a)}{T(r,f)}.$$

Theorem 12.4. *Let f be meromorphic. Then the set $A = \{a \in \widehat{\mathbb{C}} \colon \Theta(a,f) > 0\}$ is countable and*

$$\sum_{a\in\widehat{\mathbb{C}}} \Theta(a,f) := \sum_{a\in A} \Theta(a,f) \leq 2.$$

Corollary. For a meromorphic function f we have

$$\sum_{a\in\widehat{\mathbb{C}}} (\delta(a,f) + \varepsilon(a,f)) \leq 2$$

and, in particular,

$$\sum_{a\in\widehat{\mathbb{C}}} \delta(a,f) \leq 2.$$

The second inequality in the corollary is called the *deficiency relation.*

Proof of Theorem 12.4. Let $a_1, \ldots, a_q \in \widehat{\mathbb{C}}$. Dividing the inequality

$$(q-2)T(r,f) \leq \sum_{j=1}^{q} \overline{N}(r,a_j) + S(r,f)$$

from Theorem 12.3 by $T(r,f)$ and taking a limit as $r \to \infty$ through a sequence of r-values outside the exceptional set E we get

$$q - 2 \leq \sum_{j=1}^{q} \limsup_{r\to\infty} \frac{\overline{N}(r,a_j)}{T(r,f)}$$

and thus

$$\sum_{j=1}^{q} \Theta(a_j, f) \leq 2.$$

This implies that the set

$$A_q = \left\{ a \in \mathbb{C}\colon \ \Theta(a, f) \geq \frac{2}{q} \right\}$$

has at most q elements. In particular, A_q is finite and thus $A = \bigcup_{q\in\mathbb{N}} A_q$ is countable. Writing $A = \{a_1, a_2, a_3, \dots\}$ we obtain the conclusion. □

Remark. Let f be a transcendental meromorphic function. Suppose that f takes the values $a_1, \dots, a_q$ only finitely often. Then $\delta(a_j) = 1$ for all j and thus $q \leq 2$ by the deficiency relation. Thus Picard's theorem follows from the deficiency relation.

Theorem 12.5. *Let f be a non-constant meromorphic function, $a_1, \dots, a_q \in \widehat{\mathbb{C}}$ distinct and $m_1, \dots, m_q \in \mathbb{N}$. Suppose that all a_j-points of f have multiplicity at least m_j, for $j = 1, \dots, q$. Then*

$$\sum_{j=1}^{q} \left(1 - \frac{1}{m_j}\right) \leq 2.$$

Proof. By hypothesis we have

$$N(r, a_j) \geq m_j \overline{N}(r, a_j).$$

Together with the first fundamental theorem we obtain

$$\overline{N}(r, a_j) \leq \frac{1}{m_j} N(r, a_j) \leq \frac{1}{m_j} T(r, f) + \mathcal{O}(1)$$

and thus

$$\Theta(a_j, f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, a_j)}{T(r, f)} \geq 1 - \frac{1}{m_j}.$$

The conclusion now follows from Theorem 12.4. □

Remark. If f does not take a value a_j at all, that is, $f(z) \neq a_j$ for all $z \in \mathbb{C}$, then the hypothesis of Theorem 12.5 is satisfied for any $m_j \in \mathbb{N}$.

Taking the limit as $m_j \to \infty$ we may put $1/m_j = 0$ in Theorem 12.5 then.

Example. Let $f(z) = \cos z$, $a_1 = 1$, $a_2 = -1$, $a_3 = \infty$. We can take $m_1 = m_2 = 2$ and, in the sense of the previous remark, $m_3 = \infty$ and $1/m_3 = 0$. Then we have equality in Theorem 12.5.

Definition 12.2. Let f be meromorphic and $a \in \widehat{\mathbb{C}}$. Then a is called *totally ramified* if f has no simple a-points.

Theorem 12.6. *A non-constant meromorphic function has at most four totally ramified values.*

A non-constant entire function has at most two totally ramified values in $\mathbb{C}$.

Proof. Let $a_1, \ldots, a_q$ be totally ramified values. With $m_1 = m_2 = \ldots = m_q = 2$ we deduce from Theorem 12.5 that $q \cdot \frac{1}{2} \leq 2$ and thus $q \leq 4$.

If f is entire, we may choose $a_q = \infty$ and $m_q = \infty$. Applying Theorem 12.5 with $m_1 = m_2 = \ldots = m_{q-1} = 2$ yields that $q - 1 \leq 2$, so there are at most 2 totally ramified values in $\mathbb{C}$. □

The example $f(z) = \cos z$ considered above shows that an entire function may have two totally ramified values in $\mathbb{C}$. There are also meromorphic functions with four totally ramified values. They come from the theory of elliptic (doubly periodic) functions.

13 Uniqueness theorems

Our first application of the second fundamental theorem are some uniqueness theorems.

Definition 13.1. Let f and g be meromorphic and $a \in \widehat{\mathbb{C}}$. Then f and g are said to *share* the value a if $f^{-1}(a) = g^{-1}(a)$; that is, for $z \in \mathbb{C}$ we have $f(z) = a$ if and only if $g(z) = a$.

If, in addition, for every a-point z_0 of f and hence g the multiplicity of z_0 as an a-point of f is the same as the multiplicity as an a-point of g, then f and g are said to share the value a *with multiplicity*.

Example. (1) Let

$$f(z) = e^z \quad \text{and} \quad g(z) = e^{-z} = \frac{1}{f(z)}.$$

Then f and g share 0, ∞, 1 and -1 with multiplicity.

(2) Let

$$f(z) = \frac{e^z + 1}{(e^z - 1)^2} \quad \text{and} \quad g(z) = \frac{(e^z + 1)^2}{8(e^z - 1)}$$

Then f and g share 0, ∞, 1 and $-\frac{1}{8}$, but without multiplicity. This is clear for the values 0 and ∞. Moreover, we have

$$f(z) = 1 \;\Leftrightarrow\; e^z + 1 = (e^z - 1)^2 \;\Leftrightarrow\; e^z = e^{2z} - 2e^z \;\Leftrightarrow\; e^z = 3$$

and

$$\begin{aligned} g(z) = 1 \;&\Leftrightarrow\; (e^z + 1)^2 = 8(e^z - 1) \;\Leftrightarrow\; e^{2z} + 2e^z + 1 = 8e^z - 8 \\ &\Leftrightarrow\; e^{2z} - 6e^z + 9 = 0 \;\Leftrightarrow\; (e^z - 3)^2 = 0 \;\Leftrightarrow\; e^z = 3. \end{aligned}$$

Thus f and g share 1. An analogous argument shows that they also share $-\frac{1}{8}$.

Theorem 13.1. *Let f and g be meromorphic and non-constant. If f and g share 5 values, then $f = g$.*

Proof. Let $a_1 \ldots, a_5$ be the values shared by f and g. Without loss of generality we may assume that $a_1 \ldots, a_5 \in \mathbb{C}$, since otherwise we can consider $1/(f-c)$ and $1/(g-c)$ instead of f and g for a suitable $c \in \mathbb{C}$.

Put $h = f - g$. We have to show that $h = 0$. Suppose that this is not the case. Picard's theorem says that f has an a_j-point for some value of j. This a_j-point is a zero of h. Thus h is non-constant.

For $j \in \{1, \ldots, 5\}$ we put

$$\overline{N}(r, a_j) = \overline{N}\left(r, \frac{1}{f - a_j}\right).$$

Since f and g share the values $a_1 \ldots, a_5$, we also have

$$\overline{N}(r, a_j) = \overline{N}\left(r, \frac{1}{g - a_j}\right).$$

The first fundamental theorem yields that

$$\sum_{j=1}^{5} \overline{N}(r, a_j) \le \overline{N}\left(r, \frac{1}{h}\right) \le T(r,h) + \mathcal{O}(1) \le T(r,f) + T(r,g) + \mathcal{O}(1).$$

On the other hand, by the second fundamental theorem (more precisely, Theorem 12.3) we have

$$3T(r,f) \le \sum_{j=1}^{5} \overline{N}(r, a_j) + S(r,f)$$

with $S(r,f)$ as in Theorem 12.1. Thus

$$T(r,f) \le \left(\frac{1}{3} + o(1)\right) \sum_{j=1}^{5} \overline{N}(r, a_j) \quad \text{for } r \notin E_f,$$

with a subset E_f of $[0, \infty)$ of finite measure. Analogously,

$$T(r,g) \le \left(\frac{1}{3} + o(1)\right) \sum_{j=1}^{5} \overline{N}(r, a_j) \quad \text{for } r \notin E_g,$$

with a subset E_g of $[0, \infty)$ of finite measure. Combining the above estimates we obtain

$$\begin{aligned} T(r,f) + T(r,g) &\le \left(\frac{2}{3} + o(1)\right) \sum_{j=1}^{5} \overline{N}(r, a_j) \\ &\le \left(\frac{2}{3} + o(1)\right) (T(r,f) + T(r,g)) + \mathcal{O}(1) \end{aligned}$$

for $r \notin E_f \cup E_g$. This is a contradiction. □

Remark. (1) The examples given before Theorem 13.1 show that the number 5 in Theorem 13.1 cannot be replaced by 4.

(2) If f and g share 4 values $a_1, \ldots, a_4$, the arguments in the proof of Theorem 13.1 yield that

$$\sum_{j=1}^{4} \overline{N}(r, a_j) \leq T(r, f) + T(r, g) + \mathcal{O}(1),$$

$$T(r, f) \leq \left(\frac{1}{2} + o(1)\right) \sum_{j=1}^{4} \overline{N}(r, a_j) \quad \text{for } r \notin E_g$$

and

$$T(r, g) \leq \left(\frac{1}{2} + o(1)\right) \sum_{j=1}^{4} \overline{N}(r, a_j) \quad \text{for } r \notin E_g.$$

With $E = E_f \cup E_g$ this yields that

$$T(r, f) \sim T(r, g) \quad \text{for } r \notin E$$

as well as

$$\sum_{j=1}^{4} \overline{N}(r, a_j) \sim 2T(r, f) \sim 2T(r, g) \quad \text{for } r \notin E.$$

Theorem 13.2. *Let f and g be meromorphic and non-constant. If f and g share 4 values counting multiplicities, then there exists a Möbius transformation M such that $f = M \circ g$.*

Proof. Without loss of generality we may assume that 0, 1 and ∞ are among the values shared by f and g. Let c be the fourth value shared by f and g.

Let E_f and E_g be the exceptional sets arising from the second fundamental theorem, (which in turn arise from the lemma of the logarithmic derivative), applied to f and g, and put $E = E_f \cup E_g$.

We note here that the exceptional in the second fundamental theorem does not depend on the choice of the values $a_1, \ldots, a_q$. To see this recall that the exceptional set arises from the application of Borel's lemma (Lemma 6.1) in the proof of the lemma on the logarithmic derivative. More precisely, this lemma was used to estimate the term $T(R, f)$ occurring in Theorem 11.4 by $KT(r, f)$ with a constant K. In the proof of the second fundamental theorem we also have to estimate the terms $m(r, f'/(f - a_j))$. Theorem 11.4 gives an upper bound in terms of $T(R, f - a_j)$. However, we have $T(R, f - a_j) \leq T(R, f) + \log^+ |a_j| + \log 2$ by Theorem 5.1, (*ii*). This gives an estimate of $m(r, f'/(f - a_j))$ in terms of $T(R, f)$. Hence we obtain an estimate of $m(r, f'/(f - a_j))$ in terms of $T(r, f)$ outside the same exceptional set as before.

Thus, by the second fundamental theorem, there exist at most two values $a \in \widehat{\mathbb{C}}$ such that

$$\overline{N}(r, a) = o(T(r, f)) \quad \text{for } r \notin E.$$

So at least two of the shared values do not have this property, and we may assume without loss of generality that this is the case for 0 and ∞; that is,

$$\limsup_{\substack{r\to\infty\\ r\notin E}} \frac{\overline{N}\left(r,\dfrac{1}{f}\right)}{T(r,f)} > 0 \quad \text{and} \quad \limsup_{\substack{r\to\infty\\ r\notin E}} \frac{\overline{N}(r,f)}{T(r,f)} > 0$$

We consider the auxiliary function

$$H := \frac{f'}{f(f-1)(f-c)} - \frac{g'}{g(g-1)(g-c)}.$$

We want to show that $H = 0$ and thus assume that this is not the case. We have the partial fraction decomposition

$$\frac{f'}{f(f-1)(f-c)} = \alpha\frac{f'}{f} + \beta\frac{f'}{f-1} + \gamma\frac{f'}{f-c},$$

with $\alpha = 1/c$, $\beta = 1/(1-c)$ and $\gamma = 1/(c(c-1))$. The lemma on the logarithmic derivative (Theorem 11.5) implies that

$$m\left(\frac{f'}{f(f-1)(f-c)}\right) = o(T(r,f)) \quad \text{for } r \notin E.$$

Analogously we have

$$m\left(\frac{g'}{g(g-1)(g-c)}\right) = o(T(r,g)) \quad \text{for } r \notin E.$$

As remarked before Theorem 13.2, we have

$$T(r,f) \sim T(r,g) \quad \text{for } r \notin E.$$

Altogether we thus obtain

$$m(r,H) = o(T(r,f)) \quad \text{for } r \notin E.$$

Next we show that H is entire. In order to do this we note that poles of H can only occur at points where f and g take one of the shared values 0, 1, c and ∞. Using the partial fraction decomposition mentioned above we have

$$H = \alpha\left(\frac{f'}{f} - \frac{g'}{g}\right) + \beta\left(\frac{f'}{f-1} - \frac{g'}{g-1}\right) + \gamma\left(\frac{f'}{f-c} - \frac{g'}{g-c}\right),$$

If z_0 is a zero of f and hence g of multiplicity p, then f'/f and g'/g have simple poles with residue p at z_0. This implies that H is holomorphic at z_0. The same argument shows that H is holomorphic at the 1-points and c-points of H.

Suppose now that z_0 is a pole of multiplicity p. Then f' has a pole of multiplicity $p+1$ at z_0 while $f(f-1)(f-c)$ has a pole of multiplicity $3p$ at z_0. Thus the function $f'/(f(f-1)(f-c))$ has a zero (of multiplicity $2p-1$) at z_0, and so does $g'/(g(g-1)(g-c))$. It follows that H has a zero at z_0. In particular, H is entire as claimed.

Moreover, the argument shows that

$$\overline{N}(r,f) \le N\left(r, \frac{1}{H}\right).$$

Together with the first fundamental theorem we thus have

$$\overline{N}(r,f) \le T(r,H) + \mathcal{O}(1).$$

Since H is entire we have $N(r,H)=0$ and thus the estimate for $m(r,H)$ obtained above yields

$$T(r,H) = m(r,H) = o(T(r,f)) \quad \text{for } r \notin E.$$

Combining the last two equations we have

$$\overline{N}(r,f) = o(T(r,f)) \quad \text{for } r \notin E.$$

However, we assumed that

$$\limsup_{\substack{r\to\infty \\ r\notin E}} \frac{\overline{N}(r,f)}{T(r,f)} > 0.$$

This is a contradiction. Thus $H = 0$.

Next we define a second auxiliary function

$$K := \frac{f'f}{(f-1)(f-c)} - \frac{g'g}{(g-1)(g-c)}.$$

Arguments analogous to the ones used above show that K is entire and, assuming $K \ne 0$,

$$\overline{N}\left(r, \frac{1}{f}\right) \le N\left(r, \frac{1}{K}\right) \le T(r,K) + \mathcal{O}(1)$$

as well as

$$T(r,K) = m(r,K) = o(T(r,f)) \quad \text{for } r \notin E.$$

This contradicts the assumption that

$$\limsup_{\substack{r\to\infty \\ r\notin E}} \frac{\overline{N}\left(r, \frac{1}{f}\right)}{T(r,f)} > 0.$$

Thus also $K = 0$. Combining this with the equation $H = 0$ obtained already we find that

$$\begin{aligned} 0 = K &= \frac{f'f^2}{f(f-1)(f-c)} - \frac{g'g^2}{g(g-1)(g-c)} \\ &= \frac{f'}{f(f-1)(f-c)} f^2 - \frac{g'}{g(g-1)(g-c)} g^2 \\ &= \frac{f'}{f(f-1)(f-c)} (f^2 - g^2) \end{aligned}$$

and thus $f^2 = g^2$. Hence $f = g$ or $f = -g$. $\square$

Remark. The second example after Definition 13.1 shows that the conclusion of Theorem 13.2 need not hold if f and g share the values without multiplicities. However, a result of Gundersen (1983) says that it suffices to assume that 2 of the 4 values are shared counting multiplicity. It is an open question whether this also holds if only one of the 4 values is shared counting multiplicity.

14 An application to iteration theory

To motivate the application of Nevanlinna theory we are concerned with, we first give the following application of Picard's theorem.

Theorem 14.1. *Let f be an entire function which is not if the form $f(z) = z + c$ for some $c \in \mathbb{C}$. Then $f \circ f$ has a fixed point; that is, there exists $z_0 \in \mathbb{C}$ with $f(f(z_0)) = z_0$.*

Proof. Suppose that $f \circ f$ has no fixed point. Then f has no fixed point. Thus $f(z) \neq z$ for all $z \in \mathbb{C}$ and this implies that $f(f(z)) \neq f(z)$ for all $z \in \mathbb{C}$. We deduce that the function h defined by

$$h(z) = \frac{f(f(z)) - z}{f(z) - z}$$

is entire and satisfies $f(z) \neq 0$ and $f(z) \neq 1$ for all $z \in \mathbb{C}$. By Picard's theorem, h is constant, say $h(z) = c$ for all $z \in \mathbb{C}$, with $c \in \mathbb{C} \setminus \{0, 1\}$. Hence

$$f(f(z)) - z = c(f(z) - z)$$

for all $z \in \mathbb{C}$. Differentiating this we obtain

$$f'(f(z))f'(z) - 1 = c(f'(z) - 1)$$

and thus

$$(f'(f(z)) - c)f'(z) = 1 - c.$$

Since $c \neq 1$ this yields that $f'(z) \neq 0$ for all $z \in \mathbb{C}$. This implies that $f'(f(z)) \neq 0$ for all $z \in \mathbb{C}$. Moreover, the last equation implies that $f'(f(z)) \neq c$ for all $z \in \mathbb{C}$. Picard's theorem implies that $f' \circ f$ is constant. Hence f' is constant. (Note that the assumption that f has no fixed point implies that f is non-constant.) Thus f has the form $f(z) = az + b$ with $a, b \in \mathbb{C}$. The hypothesis that f has no fixed points no implies that $a = 1$. □

Remark. If f is a transcendental entire function, one would expect that $f \circ f$ has infinitely many fixed points. This is indeed the case, but it does *not* follow from Picard's theorem with the above method of proof. The point is that even if f has only one fixed point a, the equation $f(f(z)) = f(z)$ will in general have infinitely solutions and thus the function h considered in the above proof will have infinitely many 1-points. In fact, every a-point of f is a 1-point of h.

However, Nevanlinna theory can be used to show that h has "few" 1-points; that is, $\overline{N}(r,1/h) = o(T(r,h))$. The proof can then be completed as before.

We will actually prove a more general result involving the iterates f^n defined by $f^1 = f$ and $f^n = f \circ f^{n-1}$ for $n \geq 2$.

Theorem 14.2. *Let f be a transcendental meromorphic function and let g be entire. Then there exists a subset E of $[0,\infty)$ of finite measure such that*

$$\lim_{\substack{r\to\infty \\ r\notin E}} \frac{T(r, f\circ g)}{T(r,g)} = \infty$$

with some subset E of $[0,\infty)$ of finite measure.

Corollary. Let f be a transcendental entire function and $m,n \in \mathbb{N}$ with $n > m$. Then

$$\lim_{\substack{r\to\infty \\ r\notin E}} \frac{T(r, f^n)}{T(r,f^m)} = \infty$$

Proof of Theorem 14.2. Let $b \in \mathbb{C}$ be such that f has infinitely many b-points $a_1, a_2, a_3, \ldots$. Then $g(z) = a_j$ implies that $f(g(z)) = b$. For $q \in \mathbb{N}$ and $r \geq 1$ we thus obtain

$$\overline{N}\left(r, \frac{1}{f\circ g - b}\right) \geq \sum_{j=1}^{q} \overline{N}\left(r, \frac{1}{g - a_j}\right).$$

(In fact, this also holds with $\overline{N}(r,\cdot)$ replaced by $N(r,\cdot)$, but we will not need this.) By the second fundamental theorem we have

$$\sum_{j=1}^{q} \overline{N}\left(r, \frac{1}{g-a_j}\right) \geq (q-2-o(1))T(r,g) \quad \text{for } r \notin E,$$

with a subset E of $[0,\infty)$ of finite measure depending only on g. By the first fundamental theorem we have

$$\overline{N}\left(r, \frac{1}{f\circ g - b}\right) \leq T(r, f\circ g) + \mathcal{O}(1).$$

Combining the last two inequalities we obtain

$$T(r, f\circ g) \geq (q-2-o(1))T(r,g) \quad \text{for } r \notin E$$

so that

$$\liminf_{\substack{r\to\infty \\ r\notin E}} \frac{T(r, f\circ g)}{T(r,g)} \geq q-2.$$

Since $q \in \mathbb{N}$ was arbitrary, the conclusion follows. □

Remark. The conclusion of Theorem 14.2 and the Corollary actually holds without any exceptional set E.

Definition 14.1. Let f be entire, $n \in \mathbb{N}$ and $z_0 \in \mathbb{C}$. Then z_0 is called a periodic point of period n if $f^n(z_0) = z_0$ and $f^k(z_0) \neq z_0$ for $1 \leq k < n$.

The following result is due to Baker (1960).

Theorem 14.3. *Let f be a transcendental entire function. Then there exists at most one $m \in \mathbb{N}$ such that f has only finitely many periodic points of period m.*

Proof. Suppose that there are two integers $m, n \in \mathbb{N}$ such that f has only finitely many periodic points of periods m and n. Without loss of generality we may assume that $m < n$. Let $a_1, \dots, a_p$ be the periodic points of period m and $b_1, \dots, b_q$ be the periodic points of period n.

We put $\ell = n - m$ and consider the function h defined by

$$h(z) = \frac{f^n(z) - z}{f^\ell(z) - z}.$$

As in the proof of Theorem 14.1 we consider the zeros, 1-points and poles of h.

First we note that by the first fundamental theorem and Theorem 14.2 we have

$$\begin{aligned}\overline{N}(r, h) \leq \overline{N}\left(r, \frac{1}{f^\ell(z) - z}\right) &\leq T(r, f^\ell(z) - z) + \mathcal{O}(1) \\ &\leq T(r, f^\ell) + \mathcal{O}(\log r) \\ &= o(T(r, f^n))\end{aligned}$$

as $r \to \infty$, $r \notin E$. Here E is again a subset of $[0, \infty)$ of finite measure.

Let $z_0 \in \mathbb{C}$ with $h(z_0) = 0$. Then $f^n(z_0) = z_0$. Thus $z_0 = b_j$ for some $j \in \{1, \dots, q\}$ or there exists $k \in \mathbb{N}$ with $1 \leq k < n$ such that $f^k(z_0) = z_0$. It follows that

$$\overline{N}\left(r, \frac{1}{h}\right) \leq q \log r + \sum_{k=1}^{n-1} \overline{N}\left(r, \frac{1}{f^k(z) - z}\right) + \mathcal{O}(1).$$

By the first fundamental theorem and Theorem 14.2 we thus have

$$\overline{N}\left(r, \frac{1}{h}\right) \leq q \log r + \sum_{k=1}^{n-1} T(r, f^k(z) - z) + \mathcal{O}(1) = o(T(r, f^n))$$

as $r \to \infty$, $r \notin E$.

Let now $z_0 \in \mathbb{C}$ with $h(z_0) = 1$. Then $f^\ell(z_0) = f^n(z_0) = f^m(f^\ell(z_0))$ so that $f^\ell(z_0)$ is a fixed point of f^m. Thus $f^\ell(z_0) = a_j$ for some $j \in \{1, \dots, p\}$ or there exists $k \in \mathbb{N}$ with $1 \leq k < m$ such that $f^{k+\ell}(z_0) = f^k(f^\ell(z_0)) = f^\ell(z_0)$. This implies that

$$\overline{N}\left(r, \frac{1}{h-1}\right) \leq \sum_{j=1}^{p} \overline{N}\left(r, \frac{1}{f^\ell(z) - a_j}\right) + \sum_{k=1}^{m-1} \overline{N}\left(r, \frac{1}{f^{k+\ell}(z) - f^\ell(z)}\right).$$

The first fundamental theorem yields that

$$\overline{N}\left(r,\frac{1}{h-1}\right)\leq p\cdot T(r,f^{\ell})+\sum_{k=1}^{m-1}(T(r,f^{k+\ell})+T(r,f^{\ell}))+\mathcal{O}(\log r)$$

and as before we deduce from Theorem 14.2 that

$$\overline{N}\left(r,\frac{1}{h-1}\right)=o(T(r,f^n))$$

as $r\to\infty$, $r\notin E$.

Altogether we thus have

$$\overline{N}(r,h)=o(T(r,f^n)),\quad \overline{N}\left(r,\frac{1}{h}\right)=o(T(r,f^n))\ \text{ and }\ \overline{N}\left(r,\frac{1}{h-1}\right)=o(T(r,f^n))$$

as $r\to\infty$, $r\notin E$.

Using Theorem 14.2 once more it is not difficult to deduce from the definition of h that $T(r,h)\sim T(r,f^n)$ as $r\to\infty$, $r\notin E$. Together with the previous equations this is a contradiction to the second fundamental theorem. □

Remark. The example $f(z)=e^z+z$ shows that an entire functions need not have fixed points. In other words, there need not be periodic points in period 1. Thus $m=1$ may arise as an exception in Theorem 14.2. One can show that this is the only exception; that is, for $n\geq 2$ there are infinitely many periodic points of period n.

The second fundamental theorem concerns the a-points of a meromorphic function f; that is, the zeros of $f-a$. In the above proof we used the second fundamental theorem to obtain conclusions about the zeros of $f(z)-z$. More generally, one may also consider the zeros of $f(z)-a(z)$ for certain functions a.

Theorem 14.4. *Let f be meromorphic and let a_1,a_2,a_q be meromorphic functions satisfying $T(r,a_j)=o(T(r,f))$ as $r\to\infty$, for $j=1,2,3$. Then*

$$T(r,f)\leq\sum_{j=1}^{3}\overline{N}\left(r,\frac{1}{f-a_j}\right)+o(T(r,f))$$

as $r\to\infty$, $r\notin E$, for some subset E of $[0,\infty)$ of finite measure.

Proof. We consider the meromorphic function h defined by

$$h(z)=\frac{a_3(z)-a_2(z)}{a_3(z)-a_1(z)}\cdot\frac{f(z)-a_1(z)}{f(z)-a_2(z)}.$$

The hypothesis that $T(r,a_j)=o(T(r,f))$ and the first fundamental theorem yield that

$$\begin{aligned}\overline{N}\left(r,\frac{1}{h}\right)&\leq\overline{N}\left(r,\frac{1}{f-a_1}\right)+\overline{N}\left(r,\frac{1}{a_3-a_1}\right)+\overline{N}(r,a_1)+\overline{N}(r,a_2)+\overline{N}(r,a_3)\\&\leq\overline{N}\left(r,\frac{1}{f-a_1}\right)+o(T(r,f)).\end{aligned}$$

The same reasoning shows that

$$\overline{N}(r,h) \le \overline{N}\left(r, \frac{1}{f-a_2}\right) + o(T(r,f))$$

and

$$\overline{N}\left(r, \frac{1}{h-1}\right) \le \overline{N}\left(r, \frac{1}{f-a_3}\right) + o(T(r,f)).$$

Finally, using $T(r,a_j) = o(T(r,f))$ it is not difficult to see that

$$T(r,h) \sim T(r,f).$$

The conclusion now follows from the second fundamental theorem applied to h. □

Remark. Theorem 14.4 has been extended to the case of q functions $a_1, \ldots, a_q$ satisfying $T(r,a_j) = o(T(r,f))$ by Yamanoi in 2004: given $\varepsilon > 0$ we have

$$(q-2-\varepsilon)T(r,f) \le \sum_{j=1}^{q} \overline{N}\left(r, \frac{1}{f-a_j}\right)$$

for r outside a set of finite measure. With $\overline{N}(r,\cdot)$ replaced by $N(r,\cdot)$ this had been proved before by Steinmetz in 1986.

15 Differential equations in the complex domain

We will give some application of Nevanlinna theory to the theory of differential equations in the complex domain. In order to do so, we need to compare the Nevanlinna characteristic and the other quantities of Nevanlinna theory of a meromorphic function with those of the derivatives.

Theorem 15.1. *Let f be meromorphic and $k \in \mathbb{N}$. Then, for some subset E of $[0,\infty)$ of finite measure,*

(i) $N(r,f^{(k)}) = N(r,f) + k\overline{N}(r,f) \le (k+1)N(r,f)$,

(ii) $m(r,f^{(k)}) \le m(r,f) + o(T(r,f))$ *for* $r \notin E$,

(iii) $T(r,f^{(k)}) \le (k+1)T(r,f) + o(T(r,f))$ *for* $r \notin E$.

Proof. Conclusion (i) follows immediately from the definition of $N(r,\cdot)$ and $\overline{N}(r,\cdot)$.

We will prove (ii) and (iii) by induction. The case $k=0$ is trivial. Assume now that (ii) and (iii) hold for some $k \ge 0$. Noting that

$$m(r,f^{(k+1)}) = m\left(\frac{f^{(k+1)}}{f^{(k)}} f^{(k)}\right) \le m\left(\frac{f^{(k+1)}}{f^{(k)}}\right) + m(r,f^{(k)}) + \mathcal{O}(1)$$

we deduce from the lemma on the logarithmic derivative that

$$m(r,f^{(k+1)}) \le m(r,f^{(k)}) + o(T(r,f^{(k)}))$$

for $r \notin E$. Since by induction hypothesis (iii) holds for k, we deduce that $T(r, f^{(k)}) = \mathcal{O}(T(r,f))$ for $r \notin E$ and thus

$$m(r, f^{(k+1)}) \leq m(r, f^{(k)}) + o(T(r,f))$$

for $r \notin E$. Together with the assumption that (ii) holds for k we deduce that (ii) also holds with k replaced by $k+1$. Together with (i) this implies that (iii) also holds with k replaced by $k+1$. □

As a simple but instructive example of how Nevanlinna theory is applied to differential equations is the following result.

Theorem 15.2. *Let f be meromorphic and P a polynomial. If f satisfies the differential equation $f' = P(f)$, then* $\deg(P) \leq 2$.

Proof. With $d = \deg(P)$ we deduce from Theorems 5.3 and 15.1 that

$$d \cdot T(r,f) = T(r, P \circ f) + \mathcal{O}(1) = T(r, f') + \mathcal{O}(1) \leq 2T(r,f) + o(T(r,f))$$

as $r \to \infty$, $r \notin E$. This implies that $d \leq 2$. □

Remark. A solution of $f' = 1 + f^2$ is given by $f(z) = \tan z$.

Of course, the differential equation $f' = P(f)$ can be integrated directly:

$$\int_{z_0}^{z} \zeta = \int_{z_0}^{z} \frac{f'(\zeta)}{P(f(\zeta))} \zeta = \int_{f(z_0)}^{f(z)} \frac{dw}{P(w)}.$$

Above we considered the differential equation $f' = P(f)$ with a polynomial P. Thus P has the form $P(w) = \sum_{k=0}^{d} a_k w^k$ with $a_k \in \mathbb{C}$. Now we will study the more general case where the a_k are rational functions. First we extend Theorem 5.3 to this context.

Theorem 15.3. *Let f be meromorphic and*

$$P(z,w) = \sum_{k=0}^{d} a_k(z) w^k,$$

where the a_k are rational functions and $a_d \neq 0$. Then

$$T(r, P(z, f(z))) = d \cdot T(r,f) + \mathcal{O}(\log r).$$

Proof. Choose $r_0 > 0$ such that all zeros and poles of the coefficients $a_0, \ldots, a_d$ are contained in $D(0, r_0)$. Then a pole z_0 of f of order m is a pole of $P(z, f(z))$ of order dm. This implies that there exists $M \in \mathbb{N}$ such that

$$n(r, P(z, f(z))) = d \cdot n(r,f) + M$$

for $r \geq r_0$. Hence

$$N(r, P(z, f(z))) = d \cdot N(r,f) + \mathcal{O}(\log r).$$

To obtain a corresponding estimate for the proximity function we note that there exist $c_j \in \mathbb{C}\setminus\{0\}$ and $m_j \in \mathbb{Z}$ such that

$$a_j(z) \sim c_j \cdot z^{m_j}$$

as $|z| \to \infty$. Hence

$$\left|\frac{a_j(z)}{a_d(z)}\right| \le (1+o(1)) \left|\frac{c_j}{c_d}\right| \cdot |z|^{m_j - m_d}$$

as $|z| \to \infty$. Thus there exist $R > 0$ and $L > 0$ such that

$$\left|\frac{a_j(z)}{a_d(z)}\right| \le |z|^L \quad \text{and} \quad \frac{1}{|z|^d} \le |a_d(z)| \le |z|^L$$

for $|z| \ge R$. For $r \ge R$ we consider

$$J = \left\{\theta \in [0, 2\pi] \colon |f(re^{i\theta})| \ge r^{L+1}\right\}.$$

For $z = re^{i\theta}$ with $\theta \in J$ and $r \ge R$ we then have

$$\begin{aligned} |P(z, f(z))| &= \left| a_d(z) f(z)^d \left(1 + \sum_{k=0}^{d-1} \frac{a_k(z)}{a_d(z)} f(z)^{k-d}\right)\right| \\ &\ge |a_d(z)| \cdot |f(z)|^d \left(1 - \sum_{k=0}^{d-1} \left|\frac{a_k(z)}{a_d(z)}\right| |f(z)|^{k-d}\right) \\ &\ge |z|^{-L} |f(z)|^d \left(1 - \sum_{k=0}^{d-1} |z|^L |z|^{(k-d)(L+1)}\right) \\ &\ge |z|^{-L} |f(z)|^d \left(1 - \frac{d}{|z|}\right). \end{aligned}$$

For $r \ge \max\{R, 2d\}$ and $\theta \in J$ we thus have

$$|P(re^{i\theta}, f(re^{i\theta}))| \ge \frac{1}{2} r^{-L} |f(re^{i\theta})|^d.$$

Hence

$$\begin{aligned} \frac{1}{2\pi}\int_J \log^+ |P(re^{i\theta}, f(re^{i\theta}))| d\theta &\ge \frac{1}{2\pi}\int_J \left(-\log 2 - L\log r + d\log^+ |f(re^{i\theta})|\right) d\theta \\ &= d \cdot \frac{1}{2\pi}\int_J \log^+ |f(re^{i\theta})| d\theta - \log 2 - L \log r. \end{aligned}$$

Similarly,

$$|P(re^{i\theta}, f(re^{i\theta}))| \le 2r^L |f(re^{i\theta})|^d.$$

for $r \ge \max\{R, 2d\}$ and $\theta \in J$ and thus

$$\frac{1}{2\pi}\int_J \log^+ |P(re^{i\theta}, f(re^{i\theta}))| d\theta \le d \cdot \frac{1}{2\pi}\int_J \log^+ |f(re^{i\theta})| d\theta + \log 2 + L \log r.$$

For $r \geq R$ and $\theta \in [0, 2\pi] \backslash J$ we have $|f(re^{i\theta})| \leq r^{L+1}$ and

$$
\begin{aligned}
|P(re^{i\theta}, f(re^{i\theta}))| &= \left| a_d(z) \sum_{k=0}^{d} \frac{a_k(z)}{a_d(z)} f(z)^k \right| \\
&\leq |z|^L \sum_{k=0}^{d} |z|^L |z|^{(L+1)k} \leq (d+1)|z|^{2L+(L+1)d}
\end{aligned}
$$

so that

$$\frac{1}{2\pi} \int_{[0,2\pi]\backslash J} \log^+ |f(re^{i\theta})| d\theta \leq (L+1) \log r$$

and

$$\frac{1}{2\pi} \int_{[0,2\pi]\backslash J} \log^+ |P(re^{i\theta}, f(re^{i\theta}))| d\theta \leq (2L + (L+1)d) \log r + \log(d+1).$$

Combining the last estimates we deduce that

$$
\begin{aligned}
m(r, P(z, f(z))) &= \frac{1}{2\pi} \int_J \log^+ |P(re^{i\theta}, f(re^{i\theta}))| d\theta + \mathcal{O}(\log r) \\
&= d \cdot \frac{1}{2\pi} \int_J \log^+ |f(re^{i\theta})| d\theta \, \mathcal{O}(\log r) \\
&= d \cdot m(r, f) + \mathcal{O}(\log r).
\end{aligned}
$$

Together with the estimate for $N(r, \cdot)$ obtained already this yields

$$T(r, P(z, f(z))) = d \cdot T(r, f) + \mathcal{O}(\log r). \qquad \square$$

Let

$$P(z, w) = \sum_{k=0}^{d} a_k(z) w^k,$$

be as in Theorem 15.3. We say that $P(z, w)$ is a polynomial in w with rational coefficients and call d the degree with respect to w. We denote it by $\deg_w(P)$.

These polynomials form a ring. The units in this ring are the polynomials of degree 0 (in w); that is, the rational functions (of z). The corresponding field of fractions (German: Quotientenkörper) consists of functions R of the form

$$R(z, w) = \frac{\sum_{k=0}^{m} a_k(z) w^k}{\sum_{k=0}^{n} b_k(z) w^k}$$

with rational functions $a_0, \ldots, a_m$ and $b_0, \ldots, b_n$, and $b_k \neq 0$ for some k. We may actually assume that the a_k and b_k are polynomials, as this can be achieved by multiplying them with a common multiple of their denominators. In other words, R is a rational function in two variables, which means that R is a quotient of polynomials in two variables. If m and n in the above representation are chosen minimal, which means that numerator and denominator do not have a common factor (with respect to w), then $d := \max\{m, n\}$ is called the degree of R with respect to w and denoted by $\deg_w(R)$.

In elementary number theory, Bezout's lemma says that the greatest common divisor $\gcd(p, q)$ of two integers p and q has a representation $ap + bq = \gcd(p, q)$ with integers a and b. (The proof is obtained from the Euclidean algorithm.) In particular, if p and q are coprime (German: teilerfremd), then there exist a and b such that $ap + bq = 1$. This result actually holds not only for the ring of integers, but in any principal ideal domain (German: Hauptidealring). In particular, this is true for the ring of polynomials in one variable over some field. We shall use this for the field of rational functions.

Lemma 15.1. *Let $P(z, w)$ and $Q(z, w)$ be two coprime polynomials in w with rational functions in z as coefficients. Then there exist polynomials $A(z, w)$ and $B(z, w)$ in w with rational functions in z as coefficients such that*

$$A(z, w)P(z, w) + B(z, w)Q(z, w) = 1.$$

Theorem 15.4. *Let f be meromorphic and let $R(z, w)$ be rational in both variables. Then*

$$T(r, R(z, f(z))) = \deg_w(R) \cdot T(r, f) + \mathcal{O}(\log r).$$

Proof. We write

$$R(z, w) = \frac{P(z, w)}{Q(z, w)} \quad \text{with} \quad P(z, w) = \sum_{k=0}^{m} a_k(z)w^k \quad \text{and} \quad Q(z, w) = \sum_{k=0}^{n} b_k(z)w^k$$

with rational functions $a_0, \ldots, a_m, b_0, \ldots, b_n$, where $a_m \neq 0$ and $b_n \neq 0$. Assuming that P and Q are coprime we have $\deg_w(R) = \max\{m, n\}$. We first show that

$$T(r, R(z, f(z))) \leq \deg_w(R) \cdot T(r, f) + \mathcal{O}(\log r).$$

by induction on $\deg_w(R)$. The conclusion is clear if $\deg_w(R) = 0$. Suppose now that $d \in \mathbb{N}$ and that the conclusion holds if $\deg_w(R) \leq d - 1$. We may assume that $d = m > n$. In fact, if $m < n$ this can be achieved by considering $1/R(z, w)$ instead of $R(z, w)$, noting that

$$T\left(r, \frac{1}{R(z, f(z))}\right) = T(r, R(z, f(z))) + \mathcal{O}(1).$$

And if $m = n = d$, then we consider

$$\begin{aligned}\left(R(z, w) - \frac{a_d(z)}{b_d(z)}\right)^{-1} &= \left(\frac{\sum_{k=0}^{d-1}\left(a_k(z) - \frac{a_d(z)}{b_d(z)}b_k(z)\right)w^k}{\sum_{k=0}^{d} b_k(z)w^k}\right)^{-1} \\ &= \frac{\sum_{k=0}^{d} b_k(z)w^k}{\sum_{k=0}^{d-1}\left(a_k(z) - \frac{a_d(z)}{b_d(z)}b_k(z)\right)w^k}\end{aligned}$$

instead of $R(z,w)$, noting that

$$T\left(r, \left(R(z,f(z)) - \frac{a_d(z)}{b_d(z)}\right)^{-1}\right) = T(r, R(z,f(z))) + \mathcal{O}(\log r).$$

Thus we may assume that $d = m > n$. By long division (German: Polynomdivision) we find that $R(z,w)$ has the form

$$R(z,w) = \sum_{k=0}^{m-n} c_k(z) w^k + \frac{\sum_{k=0}^{\ell} d_k(z) w^k}{\sum_{k=0}^{n} b_k(z) w^k}$$

where $0 \le \ell < n$ and $c_0, \ldots, c_{m-n}, d_0, \ldots, d_\ell$ are rational functions. Thus

$$R(z, f(z)) = \alpha(z) + \beta(z)$$

where

$$\alpha(z) = \sum_{k=0}^{m-n} c_k(z) f(z)^k \quad \text{and} \quad \beta(z) = \frac{\sum_{k=0}^{\ell} d_k(z) f(z)^k}{\sum_{k=0}^{n} b_k(z) f(z)^k}.$$

By Theorem 15.3 we have $T(r,\alpha) = (m-n)T(r,f) + \mathcal{O}(\log r)$ and by the induction hypothesis we have $T(r,\beta) \le n \cdot T(r,f) + \mathcal{O}(\log r)$. Combining the two estimates we obtain the desired upper bound for $T(r, R(z, f(z)))$.

Now we turn to the proof of the lower bound for $T(r, R(z, f(z)))$; that is, we prove that

$$T(r, R(z,f(z)) \ge \deg_w(R) \cdot T(r,f) + \mathcal{O}(\log r).$$

Again we may assume that $m > n$. Since the case $n = 0$ is covered by Theorem 15.3 we may thus assume that $1 \le n < m$. Choose A and B according to Lemma 15.1 so that

$$A(z,w)P(z,w) + B(z,w)Q(z,w) = 1.$$

With $s = \deg_w(A)$ and $t = \deg_w(B)$ we have $s + m = t + n$. Since $m > n$ this yields $t > s$. We put $C = A/B$ and write $A^*(z) = A(z, f(z))$, $B^*(z, f(z))$, etc. In particular, $R^*(z) = R(z, f(z))$. Then

$$C^* + \frac{1}{R^*} = \frac{A^*}{B^*} + \frac{Q^*}{P^*} = \frac{1}{B^* Q^*}$$

and hence

$$\begin{aligned} T\left(r, C^* + \frac{1}{R^*}\right) &= T\left(r, \frac{1}{B^* Q^*}\right) \\ &= T(r, B^* Q^*) + \mathcal{O}(1) \\ &= \deg_w(BQ) \log r + \mathcal{O}(\log r) \\ &= (t+m)T(r,f) + \mathcal{O}(\log r) \end{aligned}$$

by the first fundamental theorem and Theorem 15.3. Moreover,

$$\begin{aligned} T\left(r, C^* + \frac{1}{R^*}\right) &\le T(r, C^*) + T(r, R^*) + \mathcal{O}(1) \\ &\le \deg_w(C) \cdot T(r, f) + T(r, R^*) + \mathcal{O}(\log r) \\ &= t \cdot T(r, f) + T(r, R^*) + \mathcal{O}(\log r) \end{aligned}$$

by the upper bound proved already. Combining the last two estimates we have

$$(t + m)T(r, f) = T\left(r, C^* + \frac{1}{R^*}\right) + \mathcal{O}(\log r) \le t \cdot T(r, f) + T(r, R^*) + \mathcal{O}(\log r)$$

and hence

$$\deg_w(R) \cdot T(r, f) = m \cdot T(r, f) \le T(r, R^*) + \mathcal{O}(\log r) = T(r, R(z, f(z))) + \mathcal{O}(\log r)$$

as claimed. ☐

Theorem 15.5 (Malmquist's theorem)**.** *Let f be a transcendental meromorphic function and let $R(z, w)$ be rational in both variables. Suppose that f satisfies the differential equation*

$$f'(z) = R(z, f(z)).$$

Then R is a polynomial in w and $\deg_w(R) \le 2$.

Proof. By Theorem 15.1 we have

$$T(r, f') \le 2T(r, f) + o(T(r, f))$$

for $r \notin E$ while Theorem 15.4 says that

$$T(r, R(z, f(z))) = \deg_w(R) \cdot T(r, f) + \mathcal{O}(\log r).$$

Together these results imply that $\deg_w(R) \le 2$.

For $c \in \mathbb{C}$ we consider $h = 1/(f - c)$. Then

$$h' = -\frac{f'}{(f - c)^2} = -h^2 f'.$$

Since $f(z) = 1/h(z) + c$ we find that h satisfies the differential equation

$$h'(z) = -h(z)^2 R\left(z, \frac{1}{h(z)} + c\right).$$

Applying the result we have proved already we deduce that the function S given by

$$S(z, w) = w^2 R\left(z, \frac{1}{w} + c\right)$$

satisfies $\deg_w(S) \le 2$. From this it is not difficult to deduce that R is a polynomial in w. ☐

The differential equation

$$f'(z) = a_0(z) + a_1(z)f(z) + a_2(z)f(z)^2$$

is called *Riccati differential equation.* If a_0, a_1 and a_2 are polynomials, then all solutions of this equation are meromorphic.

We only sketch the proof. Consider the following linear system of differential equations:

$$\begin{aligned} u' &= a_1 u + a_0 v \\ v' &= -a_2 u \end{aligned}$$

Standard techniques show that for every linear system of differential equations with entire coefficients the solutions are also entire. Thus all solutions u and v of the above system are entire. The quotient $f = u/v$ of two solutions is thus meromorphic, and we have

$$f' = \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2} = \frac{(a_1 u + a_0 v)v - u(-a_2 u)}{v^2} = a_0 + a_1 f + a_2 f^2.$$

The following result is proved in the same way as the previous theorem.

Theorem 15.6 (Malmquist-Yosida theorem)**.** *Let f be a transcendental meromorphic function, $R(z,w)$ be rational in both variables and $n \in \mathbb{N}$. Suppose that f satisfies the differential equation*

$$f'(z)^n = R(z, f(z)).$$

Then R is a polynomial in w and $\deg_w(R) \leq 2n$*.*

Remark. Actually there are much stronger restrictions on R than just $\deg_w(R) \leq 2n$ It was shown by Steinmetz as well as Bank and Kaufman in 1980 that the differential equation considered can be reduced by linear transformations to one of the following equations (or a power thereof):

$$\begin{aligned} f'(z) &= a(z) + b(z)f(z) + c(z)f(z)^2 \\ f'(z)^2 &= a(z)(f(z) - b(z))^2(f(z) - \tau_1)(f(z) - \tau_2) \\ f'(z)^2 &= a(z)(f(z) - \tau_1)(f(z) - \tau_2)(f(z) - \tau_3)(f(z) - \tau_4) \\ f'(z)^3 &= a(z)(f(z) - \tau_1)^2(f(z) - \tau_2)^2(f(z) - \tau_3)^2 \\ f'(z)^4 &= a(z)(f(z) - \tau_1)^2(f(z) - \tau_2)^3(f(z) - \tau_3)^3 \\ f'(z)^6 &= a(z)(f(z) - \tau_1)^3(f(z) - \tau_2)^4(f(z) - \tau_3)^5. \end{aligned}$$

There the τ_j are distinct constants and a, b and c are rational functions.

Another generalization of Theorem 15.6 due to Eremenko says that if

$$f'(z)^m Q_m(z, f(z)) + \cdots + f'(z)Q_1(z, f(z)) + Q_0(z, f(z)) = 0$$

with polynomials $Q_j(z, w)$ in w, then $\deg_w(Q_j) \leq 2(m - j)$.

16 Exceptional values of derivatives

Picard's theorem says that if an entire function f satisfies $f(z) \neq 0$ and $f(z) \neq 1$ for all $z \in \mathbb{C}$, then f is constant. Saxer showed in 1923 that this also holds if $f(z) \neq 0$ and $f'(z) \neq 1$ for all $z \in \mathbb{C}$. We will consider some more general results. In fact, Saxer's theorem is an immediate consequence of the following theorem.

Theorem 16.1. *Let f be a transcendental meromorphic function and $k \in \mathbb{N}_0$. Let $N_0(r)$ denote the counting function of the zeros of $f^{(k+1)}$ that are not 1-points of f. Then*

$$T(r,f) \leq \overline{N}(r,f) + N\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f^{(k)}-1}\right) - N_0(r) + o(T(r,f))$$

as $r \to \infty$, $r \notin E$, for some subset E of $[0,\infty)$ of finite measure.

Proof. The second fundamental theorem, applied to $g := f^{(k)}$, yields that

$$m(r,g) + m\left(r,\frac{1}{g}\right) + m\left(r,\frac{1}{g-1}\right) \leq 2T(r,g) - N_1(r,g) + o(T(r,g))$$

as $r \to \infty$, $r \notin E$, with

$$N_1(r,g) = N\left(r,\frac{1}{g'}\right) + 2N(r,g) - N(r,g').$$

Here and in the following E always denotes a subset of $[0,\infty)$ of finite measure. The first fundamental theorem implies that

$$\begin{aligned}
2T(r,g) - N_1(r,g) &= m(r,g) + N(r,g) + m\left(r,\frac{1}{g-1}\right) + N\left(r,\frac{1}{g-1}\right) \\
&\quad - \left(N\left(r,\frac{1}{g'}\right) + 2N(r,g) - N(r,g')\right) + \mathcal{O}(1) \\
&= m(r,g) + m\left(r,\frac{1}{g-1}\right) + N\left(r,\frac{1}{g-1}\right) - N\left(r,\frac{1}{g'}\right) \\
&\quad + N(r,g') - N(r,g) + \mathcal{O}(1).
\end{aligned}$$

Since

$$N\left(r,\frac{1}{g-1}\right) - N\left(r,\frac{1}{g'}\right) = \overline{N}\left(r,\frac{1}{g-1}\right) - N_0(r)$$

and

$$N(r,g') - N(r,g) = \overline{N}(r,g)$$

this yields

$$2T(r,g)-N_1(r,g) = m(r,g)+m\left(r,\frac{1}{g-1}\right)+\overline{N}\left(r,\frac{1}{g-1}\right)-N_0(r)+\overline{N}(r,g)+\mathcal{O}(1).$$

Together with the inequality obtained at the beginning of the proof this yields that

$$m\left(r,\frac{1}{g}\right) \le \overline{N}\left(r,\frac{1}{g-1}\right) - N_0(r) + \overline{N}(r,g) + o(T(r,g))$$

as $r \to \infty$, $r \notin E$.

By Theorem 15.1 we have $T(r,g) = T(r,f^{(k)}) \le (k+1)T(r,f)$ as $r \to \infty$, $r \notin E$. This implies that the $o(T(r,g))$-terms occurring above can be replaced by $o(T(r,f))$. The lemma on the logarithmic derivative implies that

$$\begin{aligned} m\left(r,\frac{1}{f^{(k-1)}}\right) &= m\left(r,\frac{1}{f^{(k)}}\cdot\frac{f^{(k)}}{f^{(k-1)}}\right) \\ &\le m\left(r,\frac{1}{f^{(k)}}\right) + m\left(r,\frac{f^{(k)}}{f^{(k-1)}}\right) \\ &\le m\left(r,\frac{1}{f^{(k)}}\right) + o(T(r,f^{(k-1)}) \\ &= m\left(r,\frac{1}{f^{(k)}}\right) + o(T(r,f) \end{aligned}$$

and induction shows that

$$m\left(r,\frac{1}{f}\right) \le m\left(r,\frac{1}{f^{(k)}}\right) + o(T(r,f) = m\left(r,\frac{1}{g}\right) + o(T(r,f)$$

as $r \to \infty$, $r \notin E$. Thus

$$m\left(r,\frac{1}{f}\right) \le \overline{N}\left(r,\frac{1}{g-1}\right) + \overline{N}(r,g) - N_0(r) + o(T(r,g))$$

and hence

$$\begin{aligned} T(r,f) &= N\left(r,\frac{1}{f}\right) + m\left(r,\frac{1}{f}\right) \\ &\le N\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{g-1}\right) + \overline{N}(r,g) - N_0(r) + o(T(r,g)) \\ &= N\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f^{(k)}-1}\right) + \overline{N}(r,f) - N_0(r) + o(T(r,f)) \end{aligned}$$

as $r \to \infty$, $r \notin E$. □

Remark. The case $k = 0$ is a direct consequence of the second fundamental theorem. The theorem of Saxer mentioned above follows from the case $k = 1$.

Theorem 16.1 implies that if f is meromorphic with only finitely many zeros and poles such that $f^{(k)}$ has only finitely many 1-points for some $k \in \mathbb{N}$, then f is rational. To see this, we only have to note that Theorem 16.1 yields that $T(r,f) = \mathcal{O}(\log r)$ under these hypotheses.

One may replace the 1-points of $f^{(k)}$ by the a-points of $f^{(k)}$ for any $a \in \mathbb{C}\setminus\{0\}$. This follows by considering f/a instead of f. However, one may not replace the 1-points of $f^{(k)}$ by the zeros of $f^{(k)}$. In fact, $f(z) = e^z$ has no zeros and poles, and $f^{(k)}(z) \neq 0$ for all $z \in \mathbb{C}$ and all $k \in \mathbb{N}$. And for $f(z) = e^z + 1$ we have $f(z) \neq 1$ and $f^{(k)}(z) \neq 0$ for all $z \in \mathbb{C}$ and all $k \in \mathbb{N}$.

We will now prove a result of Hayman which implies that in order to conclude that a meromorphic function f is rational it suffices to assume that f has only finitely many zeros and $f^{(k)}$ has only finitely many 1-points for some $k \in \mathbb{N}$. So no hypothesis on the poles is required.

Theorem 16.2. *Let f be a transcendental meromorphic function and $k \in \mathbb{N}$. Then*

$$T(r,f) \leq \left(2+\frac{1}{k}\right) N\left(r,\frac{1}{f}\right) + \left(2+\frac{2}{k}\right)\overline{N}\left(r,\frac{1}{f^{(k)}-1}\right) + o(T(r,f))$$

as $r \to \infty$, $r \notin E$.

The idea is to estimate the term $\overline{N}(r,f)$ occurring on the right hand side of the inequality in Theorem 16.1 in terms of the other counting functions occuring there. In order to do so we write

$$\overline{N}(r,f) = \overline{N}_s(r,f) + \overline{N}_m(r,f),$$

where $\overline{N}_s(r,f)$ denotes the counting function of the simple poles and $\overline{N}_m(r,f)$ denotes the counting function of the multiple poles, counted without multiplicity. First we prove the following lemma.

Lemma 16.1. *Let f be a transcendental meromorphic function and let $N_0(r)$ be as in Theorem 16.1. Then*

$$k\overline{N}_s(r,f) \leq \overline{N}_m(r,f) + \overline{N}\left(r,\frac{1}{f^{(k)}-1}\right) + N_0(r) + o(T(r,f))$$

as $r \to \infty$, $r \notin E$.

Proof. Let $g = f^{(k)}$ be as in the proof of Theorem 16.1 and put

$$h(z) = \frac{g'(z)^{k+1}}{(1-g(z))^{k+2}} = \frac{\left(f^{(k+1)}(z)\right)^{k+1}}{\left(1-f^{(k)}(z)\right)^{k+2}}.$$

Let z_0 be a simple pole of f. Then $f^{(k)}$ and hence $1 - f^{(k)}$ have a pole of order $k+1$ at z_0. In fact,

$$1 - f^{(k)}(z) = \frac{a}{(z-z_0)^{k+1}} + \mathcal{O}(1) = \frac{a}{(z-z_0)^{k+1}}\left(1+\mathcal{O}\left((z-z_0)^{k+1}\right)\right)$$

as $z \to z_0$ for some non-zero constant a (depending on z_0). It follows that

$$f^{(k+1)}(z) = \frac{(k+1)a}{(z-z_0)^{k+2}} + \mathcal{O}(1) = \frac{(k+1)a}{(z-z_0)^{k+2}}\left(1+\mathcal{O}\left((z-z_0)^{k+2}\right)\right)$$

as $z \to z_0$. Hence

$$\begin{aligned} h(z) &= \frac{(k+1)^{k+1} a^{k+1}}{(z-z_0)^{(k+2)(k+1)}} \cdot \frac{(z-z_0)^{(k+1)(k+2)}}{a^{k+2}} \left(1 + \mathcal{O}\big((z-z_0)^{k+1}\big)\right) \\ &= b + \mathcal{O}\big((z-z_0)^{k+1}\big) \end{aligned}$$

as $z \to z_0$, with $b = (k+1)^{k+1}/a \in \mathbb{C}\backslash\{0\}$. Thus $h(z_0) = b \neq 0$ and $h^{(j)}(z_0) = 0$ for $1 \leq j \leq k$. Hence h/h' has a pole of multiplicity at least k at z_0. It follows that

$$\begin{aligned} k\overline{N}_s(r,f) &\leq N\left(r, \frac{h}{h'}\right) \\ &\leq T\left(r, \frac{h}{h'}\right) \\ &= T\left(r, \frac{h'}{h}\right) + \mathcal{O}(1) \\ &= N\left(r, \frac{h'}{h}\right) + m\left(r, \frac{h'}{h}\right) + \mathcal{O}(1) \\ &\leq N\left(r, \frac{h'}{h}\right) + o(T(r,h)) \\ &= \overline{N}(r,h) + \overline{N}\left(r, \frac{1}{h}\right) + o(T(r,h)) \\ &= \overline{N}(r,h) + \overline{N}\left(r, \frac{1}{h}\right) + o(T(r,f)) \end{aligned}$$

as $r \to \infty$, $r \notin E$.

Next we note that if z_0 is a pole of f of multiplicity $p \geq 2$, then h has a zero of multiplicity $(p+k)(k+2) - (p+k+1)(k+1) = p-1$ at z_0. Further zeros of h can arise only from zeros of $f^{(k+1)}$ which are not 1-points of $f^{(k)}$ so that

$$\overline{N}\left(r, \frac{1}{h}\right) \leq \overline{N}_m(r,f) + N_0(r).$$

Poles of h can arise only from zeros of $f^{(k)} - 1$ so that

$$\overline{N}(r,h) \leq \overline{N}\left(r, \frac{1}{f^{(k)} - 1}\right).$$

Combining the last three inequalities we obtain the conclusion. □

Proof of Theorem 16.2. We have

$$\overline{N}_s(r,f) + 2\overline{N}_m(r,f) \leq N(r,f)$$

and thus, together with the first fundamental theorem,

$$\overline{N}_m(r,f) \leq N(r,f) - \big(\overline{N}_s(r,f) + \overline{N}_m(r,f)\big) = N(r,f) - \overline{N}(r,f) \leq T(r,f) - \overline{N}(r,f).$$

Theorem 16.1 says that

$$T(r,f) \le \overline{N}(r,f) + N\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f^{(k)}-1}\right) - N_0(r) + o(T(r,f))$$

as $r \to \infty$, $r \notin E$. Together these inequalities yield that

$$\overline{N}_m(r,f) \le N\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f^{(k)}-1}\right) - N_0(r) + o(T(r,f))$$

as $r \to \infty$, $r \notin E$. By Lemma 16.1 we have

$$k\overline{N}_s(r,f) \le \overline{N}_m(r,f) + \overline{N}\left(r,\frac{1}{f^{(k)}-1}\right) + N_0(r) + o(T(r,f))$$

as $r \to \infty$, $r \notin E$. Combining these two estimates we obtain

$$k\overline{N}_s(r,f) \le N\left(r,\frac{1}{f}\right) + 2\overline{N}\left(r,\frac{1}{f^{(k)}-1}\right) + o(T(r,f))$$

and thus

$$\overline{N}_s(r,f) \le \frac{1}{k}N\left(r,\frac{1}{f}\right) + \frac{2}{k}\overline{N}\left(r,\frac{1}{f^{(k)}-1}\right) + o(T(r,f))$$

as $r \to \infty$, $r \notin E$. Adding the estimates for $\overline{N}_m(r,f)$ and $\overline{N}_s(r,f)$ yields that

$$\begin{aligned}\overline{N}(r,f) &= \overline{N}_m(r,f) + \overline{N}_s(r,f)\\ &\le \left(1+\frac{1}{k}\right)N\left(r,\frac{1}{f}\right) + \left(1+\frac{2}{k}\right)\overline{N}\left(r,\frac{1}{f^{(k)}-1}\right) + o(T(r,f))\end{aligned}$$

as $r \to \infty$, $r \notin E$. Inserting this in the inequality of Theorem 16.1 yields the conclusion. □

Corollary. Let f be a meromorphic function and $k \in \mathbb{N}$. Suppose that $f(z) \ne 0$ and $f^{(k)}(z) \ne 1$ for all $z \in \mathbb{C}$. Then f is constant.

To prove the corollary, we note that Theorem 16.2 implies that f is rational, say $f = P/Q$ with polynomials P and Q. Since f has no zeros, P is constant. Induction shows that $f^{(k)}$ has the form $f^{(k)} = R_k/Q^{k+1}$ with a polynomial R_k which has no common zeros with Q and satisfies $\deg R_k < (k+1)\deg Q = \deg(Q^{k+1})$. This implies that $R_k - Q^{k+1}$ has zeros and thus $f^{(k)}$ has 1-points.

Theorem 16.3. *Let g be an entire function. Suppose that $g(z) \ne 0$ and $g''(z) \ne 0$ for all $z \in \mathbb{C}$. Then g has the form $g(z) = e^{az+b}$ with constants $a, b \in \mathbb{C}$.*

Proof. We consider

$$f(z) = \frac{g(z)}{g'(z)}.$$

Then $f(z) \ne 0$ and

$$f'(z) = 1 - \frac{g(z)g''(z)}{g'(z)^2} \ne 1$$

for all $z \in \mathbb{C}$. Thus f is constant. Hence $g'(z)/g(z) = a$ for some $a \in \mathbb{C}\setminus\{0\}$. This implies that f has the form claimed. □

Remark. A result of Langley (1993) says that if g is a meromorphic function satisfying $g(z) \neq 0$ and $g''(z) \neq 0$ for all $z \in \mathbb{C}$, then g has the form $g(z) = e^{az+b}$ or the form $g(z) = 1/(az+b)^n$ for some $n \in \mathbb{N}$. The proof of this result is much more involved than that of Theorem 16.3.

References

[1] Anatoly A. Goldberg, Iossif V. Ostrovskii: Value distribution of meromorphic functions. American Mathematical Society, Providence, RI, 2008.

[2] W. K. Hayman: Meromorphic functions. Clarendon Press, Oxford 1964

[3] Gerhard Jank, Lutz Volkmann: Einführung in die Theorie der ganzen und meromorphen Funktionen mit Anwendungen auf Differentialgleichungen. Birkhäuser Verlag, Basel, 1985.

The above is only a small selection of the many books on the theory of entire and meromorphic functions.