Differential Geometry Assignment 1 - Fall 2023

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Problem 1:

Show that the arc length and curvature of a regular curve is invariant under reparametrization.

Problem 2

Determine if the curve $\tilde{\gamma}(t) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$ for $t \in \mathbb{R}$ is a reparametrization of the curve $\gamma(s) = (\cos(s), \sin(s))$. If it is, provide the reparametrization function.

Problem 3

Find a parametrization of the center of the osculating circle to the curve $\gamma(t) = (a\cos(t), b\sin(t))$ for $t \in \mathbb{R}$.

Problem 4

Show that the Frenet equations for a space curve are equivalent to the Darboux equations

$$e_i^{'} = D \times e_i$$

where $D = \tau e_1 + \kappa e_3$.

Problem 5

Prove the fundamental theorem of space curves.

Problem 6

A regular parametrized curve α has the property that all of its tangent lines pass through a fixed point. Show that

- Trace of α is a segment of straight line.
- Does the conclusion still hold if α is not regular?

- Repeat the problems if instead of tangent lines, normal lines pass through a fixed point. What can you say about the trace in this case?

Problem 7

Suppose $\alpha(s)$ is parameterized by arc-length and has the property that $\|\alpha(s)\| \leq \|\alpha(s_0)\|$ for all s sufficiently close to s_0. Show that $\kappa(s_0) \geq \frac{1}{\|\alpha(s_0)\|}$.

Problem 8

Let $\alpha : I \to \mathbb{R}$ be a regular curve such that $\kappa(t) \neq 0, \forall t$. The *Evolute* of α is the curve

$$\beta(t) = \alpha(t) + \frac{N(t)}{\kappa(t)}$$

where $N(t)$ is the Frenet normal. Show that the tangent line to β is exactly the normal line to $\alpha(t)$.

Problem 9

Let α be a regular space curve with non-zero curvature and torsion whose image lies on the unit sphere. Prove that

$$\frac{\tau}{\kappa} + (\frac{1}{\tau}(\frac{1}{\kappa})')' = 0$$

Is the converse true?

Ungraded Problems

- Use graphing software to identify curve whose curvature changes constantly with respect to the arc-length parameter. Plot the curve and provide a brief explanation of how you determined its curvature behavior.
- Prove the fundamental theorem of curves in $\mathbb{R}^n$.
- Prove that there exists a unique circle in the plane passing through three non-collinear points.