Differential Geometry (curves)

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Solve the following problems

Problem 1

Let $\alpha(s), s \in [0, l]$ be a positively oriented regular closed convex plane curve. Define

$$\beta(s) = \alpha(s) - rn(s),$$

where r is a positive constant and n is the normal vector. Show that

1. $L(\beta) = L(\alpha) + 2\pi r$
2. $\kappa_\beta(s) = \frac{\kappa_\alpha(s)}{1+r\kappa_\alpha}$

Problem 2

1. Let $\alpha(s), s \in [0, l]$ be a plane simple closed curve whose curvature satisfies $0 < \kappa(s) \leq R$, for some constant R. Prove that

$$\text{length of } \alpha \geq \frac{2\pi}{R}$$

2. What can you say about the smallest possible length of α if it has rotation index N?

Problem 3

1. Let $f : [a, b] \to \mathbb{S}^1$ be a continuous function with $f(a) = f(b)$, and let $p \in \mathbb{S}^1$. If the degree of f equals n, what is the minimal possible size of the set $\{t \in [a, b] | f(t) = p\}$?
2. Is there a simple closed curve in plane with length 6 ft. enclosing an area of 3 square feet?

Problem 4

Let C be the trace of a simple closed convex plane curve, and let L be a line. Prove that

1. If L is tangent to C at two distinct points, then C contains the entire segment of L between these two points.

2. If $C \cap L$ contains more than two points, then it contains the entire segment of L between any pair of these points.