Assignment for the course “Differential geometry”, October 9–13, 2023

Part 1

1.1. *Given a, $b>0$, the hyperbolic paraboloid is the surface given by*

$$S=\left\{(x,y,z)\in\mathbb{R}^3 \;\middle|\; \left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2-z=0\right\} .$$

Prove that the map $\varphi\colon\mathbb{R}^2\to\mathbb{R}^3$ given by

$$\varphi(x_1,x_2)=\left(a\frac{x_1+x_2}{2},b\frac{x_1-x_2}{2},x_1x_2\right)$$

is a global parametrization of S.

Solution: The set S is the graph of the C^∞ function $f(x,y)=(x/a)^2-(y/b)^2$ and thus it is a regular surface.

To prove that φ is a global parametrization of S, we have to prove that it is a local parametrization and then that $\varphi(\mathbb{R}^2)=S$. Since we already know that S is a regular surface, it suffices to prove the following three facts:

(i) $\varphi(\mathbb{R}^2)=S$;
(ii) φ is continuous and injective;
(iii) $d\varphi$ has rank 2 at all points.

(i) To prove that $\varphi(\mathbb{R}^2)\subseteq S$ take $(x_1,x_2)\in\mathbb{R}^2$. Then

$$\left(\frac{1}{a}a\frac{x_1+x_2}{2}\right)^2-\left(\frac{1}{b}b\frac{x_1-x_2}{2}\right)^2-x_1x_2=\frac{x_1^2+2x_1x_2+x_2^2}{4}-\frac{x_1-2x_1x_2+x_2^2}{4}-x_1x_2=\frac{4x_1x_2}{4}-x_1x_2=0 ,$$

and thus $\varphi(x_1,x_2)\in S$.

Conversely, take $(x,y,z)\in S$; we have to find $(x_1,x_2)\in\mathbb{R}^2$ such that $\varphi(x_1,x_2)=(x,y,z)$. First of all, this means that

$$\begin{cases} a\frac{x_1+x_2}{2}=x , \\ b\frac{x_1-x_2}{2}=y , \end{cases} \iff \begin{cases} x_1=\frac{x}{a}+\frac{y}{b} , \\ x_2=\frac{x}{a}-\frac{y}{b} . \end{cases}$$

Since $(x,y,z)\in S$ we know that $z=(x/a)^2-(y/b)^2$; hence

$$\begin{aligned}\varphi(x_1,x_2)&=\left(\frac{a}{2}\left(\frac{x}{a}+\frac{y}{b}+\frac{x}{a}-\frac{y}{b}\right),\frac{b}{2}\left(\frac{x}{a}+\frac{y}{b}-\frac{x}{a}+\frac{y}{b}\right),\left(\frac{x}{a}+\frac{y}{b}\right)\left(\frac{x}{a}-\frac{y}{b}\right)\right)\\ &=\left(\frac{a}{2}\frac{2x}{a},\frac{b}{2}\frac{2y}{b},\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2\right)=(x,y,z) ,\end{aligned}$$

as required.

(ii) Since the components of φ are polynomials, the continuity is immediate. The injectivity is easy too:

$$\varphi(x_1,x_2)=\varphi(y_1,y_2) \implies \begin{cases} a\frac{x_1+x_2}{2}=a\frac{y_1+y_2}{2} , \\ b\frac{x_1-x_2}{2}=b\frac{y_1-y_2}{2} , \end{cases} \iff \begin{cases} 2x_1=2y_1 , \\ 2x_2=2y_2 , \end{cases}$$

that is $(x_1,x_2)=(y_1,y_2)$, as desired.

(iii) The differential of φ is represented by the matrix

$$\begin{vmatrix} \frac{a}{2} & \frac{a}{2} \\ \frac{b}{2} & -\frac{b}{2} \\ x_2 & x_1 \end{vmatrix} .$$

The determinant of the upper 2×2 matrix is given by $\frac{a}{2}\left(-\frac{b}{2}\right)-\frac{b}{2}\frac{a}{2}=-\frac{ab}{2}\neq 0$, and so $d\varphi$ has always rank 2.

1.2. *Let $V \subseteq \mathbb{R}^3$ be an open subset and $f \in C^\infty(V)$. Prove that for all $a \in \mathbb{R}$ the connected components of the set $f^{-1}(a) \setminus \mathit{Crit}(f)$ are regular surfaces. Deduce that each connected component of $C \setminus \{O\}$ is a regular surface, where*

$$C = \bigl\{(x, y, z) \in \mathbb{R}^3 \bigm| x^2 + y^2 - z^2 = 0\bigr\}$$

is the double-sheeted cone.

Solution: The set $\mathrm{Crit}(f)$ of critical points of f is a closed set, because it is the intersection of the zero sets of the partial derivatives of f. Therefore $\tilde V = V \setminus \mathrm{Crit}(f)$ is an open set. Set $\tilde f = f|_{\tilde V}$; then $\tilde f \in C^\infty(\tilde V)$ has no critical points. This means, by the Proposition we proved in class, that for all $a \in \mathbb{R}$ the connected components of the set $\tilde f^{-1}(a)$ are (empty or) regular surfaces. Since $\tilde f^{-1}(a) = f^{-1}(a) \setminus \mathrm{Crit}(f)$, this shows that the connected components of $f^{-1}(a) \setminus \mathrm{Crit}(f)$ are regular surfaces, as desired.

Finally, set $f(x, y, z) = x^2 + y^2 - z^2$. Then $\nabla f = (2x, 2y, -2z)$ and thus $\mathrm{Crit}(f) = \{O\}$. Since $C = f^{-1}(0)$, the first part of the exercise implies that the connected components of $C\setminus\{O\} = f^{-1}(0)\setminus\mathrm{Crit}(f)$ are regular surfaces.

1.3. *Prove that the set*

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^3 = 1\}$$

is a regular surface and find an atlas for it.

Solution: Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be given by $f(x, y, z) = x^2 + y^2 - z^3$. Then $\nabla f = (2x, 2y, -3z^2)$ and thus $\mathrm{Crit}(f) = \{O\}$. Since $f(O) = 0$, the only critical value of f is 0; therefore 1 is a regular value and $S = f^{-1}(1)$ is a regular surface.

We describe an atlas consisting of local parametrization given by graphs of smooth functions. In this way we automatically know that they are injective, homeomorphisms with their images and that their differentials have rank 2 and we must only prove that their images cover S.

Put $U_1 = \{(x_1, x_1) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 - 1 \neq 0\}$ and let $\varphi_1\colon U_1 \to \mathbb{R}^3$ be $\varphi_1(x_1, x_2) = \bigl(x_1, x_2, (x_1^2 + x_2^2 - 1)^{1/3}\bigr)$. Clearly, $\varphi_1(U_1) \subseteq S$; more precisely, $\varphi_1(U_1) = S \setminus \{z = 0\}$. Notice that we have to remove $z = 0$ because the cubic root is not smooth at zero.

We must now cover the subset $\{(x, y, 0) \mid x^2 + y^2 = 1\} \subset S$, again using graphs. Put

$$U_2 = \{(x_1, x_2) \in \mathbb{R}^2 \mid 1 - x_1^2 + x_2^3 > 0\}$$

and let $\varphi_{2,\pm}\colon U_2 \to \mathbb{R}^3$ be $\varphi_{2,\pm}(x_1, x_2) = \bigl(\pm\sqrt{1 - x_1^2 + x_2^3}, x_1, x_2\bigr)$; notice that $\varphi_{2,\pm}$ is *not* well defined on $(-1, 1) \times \mathbb{R}$ but only on U_2. Clearly, $\varphi_{2,\pm}(U_2) \subset S$. To see which part of $S \cap \{z = 0\}$ is covered by $\varphi_{2,\pm}(U_2)$, notice that $\varphi_{2,\pm}(x_1, x_2) \in S \cap \{z = 0\}$ if and only if $x_2 = 0$; this implies $x_1^2 < 1$ and hence $\varphi_{2,+}(U_2) \cup \varphi_{2,-}(U_2)$ covers all of $S \cap \{z = 0\}$ except the two points $(0, \pm1, 0)$.

To cover the two missing points we define $\varphi_{3,\pm}\colon U_2 \to \mathbb{R}^3$ by $\varphi_{3,\pm} = \bigl(x_1, \pm\sqrt{1 - x_1^2 + x_2^3}, x_2\bigr)$. Then $\varphi_{3,\pm}(0, 0) = (0 \pm 1, 0)$ and we have covered everything. In particular,

$$\{(U_1, \varphi_1), (U_2, \varphi_{2,+}), (U_2, \varphi_{2,-}), (U_2, \varphi_{3,+}), (U_2, \varphi_{3,-})\}$$

is an atlas for S.

Part 2

2.1. *Determine a diffeomorphism between the unit sphere $S^2 \subset \mathbb{R}^3$ and the ellipsoid $E \subset \mathbb{R}^3$ of equation $4x^2 + 9y^2 + 25z^2 = 1$.*

Solution: Let $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ be given by $F(x, y, z) = \bigl(\frac{x}{2}, \frac{y}{3}, \frac{z}{5}\bigr)$. It is an invertible linear map, whose inverse is $F^{-1}(x, y, z) = (2x, 3y, 5z)$. Since the restriction of a smooth map of $\mathbb{R}^3$ to a surface is still smooth, and both F and F^{-1} are smooth in $\mathbb{R}^3$, it suffices to show that $F(S^2) = E$ to deduce that $F|_{S^2}\colon S^2 \to E$ is a diffeomorphism with inverse $F^{-1}|_E$.

To show that $F(S^2) \subseteq E$, take $(x, y, z) \in S^2$ so that $x^2 + y^2 + z^2 = 1$. Then

$$4\left(\frac{x}{2}\right)^2 + 9\left(\frac{y}{3}\right)^2 + 25\left(\frac{z}{5}\right)^2 = 4\frac{x^2}{4} + 9\frac{y^2}{9} + 25\frac{z^2}{25} = x^2 + y^2 + z^2 = 1 ,$$

that is $F(x, y, z) \in E$.

Finally, to show that $E \subseteq F(S^2)$, take $(x, y, z) \in E$. Then $(2x)^2 + (3y)^2 + (5z)^2 = 1$, that is $(2x, 3y, 5z) \in S^2$; but $F(2x, 3y, 5z) = (x, y, z)$ and we are done.

2.2. *Let $S \subset \mathbb{R}^3$ be a surface and $f \in C^\infty(S)$. Prove that if $p \in S$ is a local minimum or a local maximum for f then $df_p \equiv 0$.*

Solution: Let $\varphi: U \to S$ be a local parametrization of S centered at p. In particular, $\varphi(O) = p$ and a basis of T_pS is given by $\left\{ \frac{\partial\varphi}{\partial x_1}(O), \frac{\partial\varphi}{\partial x_2}(O) \right\}$.

Now, since p is a local minimum or a local maximum for f in S, it follows that O is a local minimum or a local maximum for $f \circ \varphi: U \to \mathbb{R}$. But this implies that $\frac{\partial(f\circ\varphi)}{\partial x_j}(O) = 0$ for $j = 1$, 2; since

$$\frac{\partial(f \circ \varphi)}{\partial x_j}(O) = df_p \left(\frac{\partial\varphi}{\partial x_j}(O) \right) ,$$

we deduce that df_p vanishes on a basis of T_pS and hence everywhere.

Warning: in general, if φ is not a local parametrization, it is not true that $d(f \circ \varphi)_O \equiv O$ implies $df_p \equiv O$. Indeed, $d(f \circ \varphi)_O = df_p \circ d\varphi_O$; therefore if the image of $d\varphi_O$ is contained in the kernel of df_p then we can have $d(f \circ \varphi)_O \equiv O$ even when $df_p \not\equiv O$. The reason why $d(f \circ \varphi)_O \equiv O$ implies $df_p \equiv O$ when φ is a local parametrization is that in this case the image of $d\varphi_O$ is the whole tangent space T_pS; therefore saying that df_p vanishes on the image of $d\varphi_O$ (this is the meaning of the formula $df_p \circ d\varphi_O \equiv O$) is equivalent to saying that df_p vanishes on the whole of T_pS and hence $df_p \equiv O$. To understand even better this argument, keep in mind that $d\varphi_O: \mathbb{R}^2 \to T_pS \subset \mathbb{R}^3$ while $dp_p: T_pS \to \mathbb{R}$.

Alternative solution: take $v \in T_pS$. By definition, there is a curve $\gamma: (-\varepsilon, \varepsilon) \to S$ such that $\gamma(0) = p$ and $\gamma'(0) = v$. The function $f \circ \gamma: (-\varepsilon, \varepsilon) \to \mathbb{R}$ has a local maximum or minimum at 0; hence $(f \circ \gamma)'(0) = 0$. But we saw that $df_p(v) = (f \circ \gamma)'(0)$; therefore $df_p(v) = 0$. Since this holds for all $v \in T_pS$ we get $df_p \equiv O$ as desired.

2.3. *Let $\varphi: (0, 2\pi) \times (0, +\infty) \to \mathbb{R}^3$ be given by*

$$\varphi(x_1, x_2) = (x_2 \cos x_1, x_2 \sin x_1, x_2^2) .$$

Prove that φ is a local parametrization of the elliptic paraboloid $S \subset \mathbb{R}^3$ given by

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 - z = 0\}$$

and then use φ to determine a basis of $T_{p_0}S$, where $p_0 = (1, 1, 2) \in S$.

Solution: We already know that S is a surface, because it is the graph of the function $f(x, y) = x^2+y^2$. Therefore to prove that φ is a local parametrization it suffices to show that $\varphi(U) \subset S$, where $U = (0, 2\pi) \times (0, +\infty)$, that φ is injective and that $d\varphi$ has rank 2 everywhere.

Since $(x_2 \cos x_1)^2 + (x_2 \sin x_1)^2 - x_2^2 = 0$, we have $\varphi(U) \subset S$. Notice that $\varphi(U)$ is not equal to S. To compute $\varphi(U)$, first of all remark that, since $x_2 > 0$ by definition, the image of φ cannot contain the point $(0, 0, 0) \in S$. If $(x, y, z) \in S$ has $z > 0$ then we can find $x_2 > 0$ such that $z = x_2^2$; moreover, $x^2 + y^2 = x_2^2$ and hence if $(x, y) \neq (x_2, 0)$ we can find $x_1 \in (0, 2\pi)$ such that $(x, y) = (x_2 \cos x_1, x_2 \sin x_1)$. Summing up, we have proved that $\varphi(U) = S \setminus \{y = 0\}$. In particular, $\varphi(U)$ is open in S because it is the intersection of S with an open set in $\mathbb{R}^3$.

Warning: $\varphi(U)$ is open in S but it is not open in $\mathbb{R}^3$, because it is not a neighbourhood in $\mathbb{R}^3$ of its points (it does not contain 3-dimensional balls).

To prove that φ is injective, first of all notice that $\varphi(x_1, x_2) = \varphi(y_1, y_2)$ implies $x_2^2 = y_2^2$ and hence $x_2 = y_2$ because they are both positive. Therefore $(\cos x_1, \sin x_1) = (\cos y_1, \sin y_1)$ and then $x_1 = y_1$ because we are assuming x_1, $y_1 \in (0, 2\pi)$.

Warning: the fact that $\cos x_1 = \cos y_1$ is not enough to conclude that $x_1 = y_1$ because cos is not injective on $(0, 2\pi)$. For the same reason, the equality $\sin x_1 = \sin y_1$ is not enough. It is the map $(\cos, \sin): (0, 2\pi) \to \mathbb{R}^2$ which is injective, not its components.

Finally, $d\varphi$ is represented by the Jacobian matrix

$$\begin{vmatrix} -x_2 \sin x_1 & \cos x_1 \\ x_2 \cos x_1 & \sin x_1 \\ 0 & 2x_2 \end{vmatrix}$$

The determinant of the upper 2×2 submatrix is $-x_2$, which is different from zero because $x_2 \in (0, +\infty)$; therefore $d\varphi$ has always rank 2 and we have proved that φ is a local parametrization.

Finally, $(1, 1, 2) = \varphi\left(\frac{\pi}{4}, \sqrt{2}\right)$; therefore a basis of $T_{p_0}S$ is given by the two columns of the Jacobian matrix of φ computed at $\left(\frac{\pi}{4}, \sqrt{2}\right)$, that is by

$$\left\{ \begin{vmatrix} -1 \\ 1 \\ 0 \end{vmatrix}, \begin{vmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 2\sqrt{2} \end{vmatrix} \right\} .$$

Warning: to get a basis of T_pS when $p = \varphi(x^o)$, the two columns of the Jacobian matrix of φ must be computed in the point x^o. They should *not* be computed in the point p, for the simple reason that p does not belong to the domain of φ and it does not make sense to compute the partial derivatives of φ in a point which is not in the domain of φ. The correct expression for the vectors of the basis of T_pS induced by φ is

$$\left.\frac{\partial}{\partial x_j}\right|_p = \frac{\partial \varphi}{\partial x_j}(x^o) \; ;$$

in the right-hand side we must have $x^o \in \mathbb{R}^2$ and not $p \in S \subset \mathbb{R}^3$.