DIFFERENTIAL GEOMETRY ASSIGNMENT
IMM, LUMS

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Exercise 1. *Explicit the intrinsic distance d_X^{int} in the case of the following subsets X of $\mathbb{R}^2$:*

- $X = \mathbb{R}^2 - \{(0,0)\}$.

- $X = \mathbb{R}^2 - B$, where B is the closed ball of center 0 and radius 1.

- $X = I(R) \times I(R) - I(r) \times I(r)$ where $I(R)$ is the interval of centre 0 and radius R (so it has length $2R$, and $I(R) \times I(R)$ is the square of center 0 and edge length $2R$. One assumes $R > R$).

Exercise 2. *Show that a regular surfaces S in $\mathbb{R}^2$ is rectifiably path-connected iff S is connected.*

Exercise 3. *(The sphere $\mathbb{S}^2$). Endow $\mathbb{S}^2$ with its intrinsic metric as a subset form $\mathbb{R}^3$.*

- Show that $\mathbb{S}^2$ is homogeneous. More precisely, the orthogonal group $O(3)$ acts transitively on it.

- Show that, more, $\mathsf{O}(3)$ acts transitively of the set of great circles.

- Show that (the rotation group) $\mathsf{SO}(3)$ (the subgroup of elements of positive determinant) too, acts transitively.

Exercise 4. *For a metric space (X, d), define an isometry f as a preserving distance bijection: $d(f(x), f(y)) = d(x, y)$).*

1. Show that any isometry f of the Euclidean plane $\mathbb{R}^2$ is affine, more precisely there exists an orthogonal matrix $A \in \mathsf{O}(2)$, and a vector T, such that $f(p) = A(p) + T$, for any $p \in \mathbb{R}^2$. (Hint: Observe that f preserves the middles of segments...).

2. Show that any isometry f of $\mathbb{S}^2$ belongs to $\mathsf{O}(3)$, that is f is either a reflection on vectorial plane $P \subset \mathbb{R}^3$, of a rotation ($\in \mathsf{SO}(3)$).

Exercise 5. *Let C be a great circle in $\mathbb{S}^2$.*

- Show that, up to isometry, we can assume $C = \mathbb{S}^1 = \mathbb{S}^2 \cap \mathbb{R}^2$.

- Describe the group of isometries preserving C, that is

$$G = \{f \in \mathsf{O}(3) \text{ such that } f(C) = C\}$$

- Show that there exists a reflection $f \in G$ fixing all points of C.

- Use this (existence of reflections) to prove that C is geodesic (Hint: use the general fact that for regular surfaces, and LOCALLY, geodesics joining close points exist and are unique).

- Actually, existence and uniqueness of geodesics for general regular surfaces, uses the differentiable side of the theory, that is geodesics satisfy an ordinary differential equations (of order two...). Is it possible to find a proof that great circles are geodesics of the sphere, without this general theory?

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Exercise 6. *Recall the notion of isometric embedding between tow metric spaces...*

Let us consider $X = \{a, b, c, d\}$ *a subset cardinality 4 of* $\mathbb{S}^2$*.*

As a starting example, let a *to be the north pole, and* b, c, d*, are on the equator. So all their distances to* a *equal* $\pi/2$*. Prove that there is no isometric embedding of* X *(endowed with the metric induced from* $\mathbb{S}^2$*) in* $\mathbb{R}^2$*.*

- More generally, such embedding do not exist unless X *is contained in a great circle. (This statement is a very strong version of the fact that there are no faithful geographical maps).*

Exercise 7. *Show that a rectifiable curve (i.e. having finite length) in a metric space admits an arc-length parametrization.*

Exercise 8. *Let* $X \subset \mathbb{R}^3$ *be a rectifiably path-connected set. Assume* X *closed (in* $\mathbb{R}^2$*). Then, show that geodesics exist, that is for any* $p, q \in X$*, with* $d_X^{int}(p, q) = l$*, there exists a curve* $c : [0, l] \to X$ *parameterized by arc-length, with* $c(0) = p, c(l) = q$*.*