DIFFERENTIAL GEOMETRY ASSIGNMENT #2
IMM, LUMS

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(Please give solutions for at least 4 exercises.)

Exercise 1. *(Surfaces of revolution). Consider a surface S of revolution determined by a curve $t \to (\alpha(t), 0, \beta(t))$, where t belongs to some interval I.*

- Compute all the geometric characteristics of S: first and second fundamental forms, area, Gaussian curvature K, mean curvature H, principal curvatures k_1, k_2, and principal directions. (Preliminary to that, show that S is orientable).

- Classify the cases where S has a constant curvature $K = 0, +1$, or -1.

Exercise 2. *Recall (and give details) why the second fundamental form of a surface S seen as a graph of a function h on its tangent plane at a point p, is the Hessian of h at p?*

- Deduce the familiar interpretation of the sign of $K(p)$ (where K is the Gaussian curvature) by means of the relative position of S with respect to its tangent plane T_pS.

Exercise 3. *Prove that a ruled surface S has a nonpositive Gaussian curvature.*

Prove that a minimal surface has a nonpositive Gaussian curvature: $K \leq 0$.

Exercise 4. *Recall the construction of the pseudo-sphere S, and show in particular that it has constant Gaussian curvature -1.*

- Show that the pseudo sphere can not be extended to a smooth surface, that if S is contained in a (regular) surface S', then $S' = S$. (Hint compute the mean curvature of S).

Exercise 5. *Let $S_c = \{(x, y, z)/x^2 + y^2 - z^2 = c\}$.*

For which value S_c is a regular surface?

Show that for $c > 0$, S_c is (doubly) ruled.

Exercise 6. *Let S be a (regular) orientable surface in $\mathbb{R}^3$. Assume S compact.*

- We want to prove that the Gauss map $N : S \to \mathbb{S}^2$ is surjective. For this, let P be a vectorial plane $\subset \mathbb{R}^3$. Consider $\pi : \mathbb{R}^3 \to P^\perp$, the orthogonal projection of $\mathbb{R}^3$ on $P^\perp$, the orthogonal line to P. Its levels are affine planes parallel to P. Identify $P^\perp$ to $\mathbb{R}$, and consider f, the restriction of π to S. - Show that if p is such that $f(p)$ is maximal or minimal, then $T_pS = P$, and moreover S lies on one side of T_pS. This means that $N(p)$ is collinear to $P^\perp$, that is $N(p) = u$, with $u \in P^\perp$. By considering both points where f is maximal and minimal, prove that , also $-u$ can be reached by N, and so $N : S \to \mathbb{S}^2$ is surjective.

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- Consider $X \subset S$, the set of points $p \in S$ such that S lies in one side of T_pS (seen as a affine plane). Show that X is closed. - Show that the image of X by the Gauss map equals $\mathbb{S}^2$. - Show that for any $p \in X$, $K(p) \geq 0$ (Use Exercise 2). - Show that for $x \in X$, $K(p) = 0$ iff p is singular for the Gauss map N.

- Sard's Theorem says that a C^∞ map between surfaces has regular values. Use it to show that there exists an elliptic point $p \in X$.

Exercise 7. *Classify ruled surfaces of revolution.*

Exercise 8. *(Geodesics of surfaces of revolution) Let S be a surface of revolution determined by a curve $m_0 : t \in [0,1] \to (\alpha(t), 0, \beta(t))$. Let $c : [a,b] \to S$ be a curve in S ($a, b \in [0,1]$), of the form: $c(t) = (\alpha(t)\cos\theta(t), \alpha(t)\sin(\theta(t), \beta(t))$. Let I be an interval contained in $[a,b] \subset [0,1]$. Show that the length of c restricted to I is greater than the length of m_0 restricted to I. Apply this to curves such that $c(a) = m_0(a), c(b) = m_0(b)$ to deduce that m_0 is a geodesic curve.*

- The curve m_0 as well as its images by rotations around the z-axis are called meridians. So, all meridians are geodesic.

- Deduce that the geodesic of the sphere are great circles.

- Prove that a meridian (more precisely its support) is a normal section of S.

Exercise 9. *From the differential point of view, a geodesic in surface S, is a curve γ for which the normal curvature (as a curve in S) coincides with its (usual) curvature as a curve in $\mathbb{R}^3$. (This definition naturally coincides with the metric one with respect to the intrinsic distance d_S^{int}, saying that the distance along γ equals the length...).*

- Show that the geodesics of an affine plane are (contained in) affine lines, and the geodesics of $\mathbb{S}^2$ are (contained in) great circles.

- Show that meridians of surface of revolution are geodesics.

- In general, say an affine plane $\mathbf{P}$ is normal to S, if for each $p \in \mathbf{P} \cap S$, the normal N_p is tangent to $\mathbf{P}$. In this case, $p \in \mathbf{P} \cap S$, is, locally (i.e. near of any of its points) a curve γ. Show that such a γ is geodesic. Show that the meridians of a surface of revolution arize in this way.

Exercise 10. *Consider $S = \mathbb{S}^2$ the unit sphere in $\mathbb{R}^3$, endowed with the canonical basis e_1, e_2, e_3 (corresponding to x, y and z-axes). Denote by R_θ^i to rotation of angle θ around the axis $\mathbb{R}e_i$. Consider the four points: $a_1 = R_\theta^3(e_2), a_2 = R_{-\theta}^3(e_2), a_3 = R_\theta^1(e_2), a_4 = R_{-\theta}^1(e_2)$ (so a_1 is the image of e_2 by a rotation of angle θ around the z-axis, ...).*

- Compute all the distances $d_{\mathbb{S}^2}^{int}(a_i, a_j)$ (for the intrinsic distance of $\mathbb{S}^2$) .

- Denote $X = \{a_1, a_2, a_3, a_4\}$ this subset of four points in $\mathbb{S}^2$ and endow it with the induced distance (form $d_{\mathbb{S}^2}^{int}$). Let Y the subset of five points $Y = X \cup \{e_2\}$.

- Show that there is no isometric embedding (i.e a distance preserving map) $X \to \mathbb{R}^2$.

- What about isometric embeddings of X or Y in higher dimensional Euclidean spaces $\mathbb{R}^n$ (especially $\mathbb{R}^3$)?