Defⁿ:- A piecewise regular curve in $\mathbb{R}^n$ is a continuous function $\gamma : [a,b] \to \mathbb{R}^n$ with a partition $a = t_0 < t_1 < \dots < t_n = b$ such that the restriction γ_i, of γ to each subintervals $[t_i, t_{i+1}]$ is a regular curve.

It is called closed if $\gamma(a) = \gamma(b)$. and simple if γ is one-to-one on the domain $[a,b)$.

It is said to be of unit speed if each γ_i is of unit speed.

A piecewise regular simple closed plane curve γ is called positively oriented if $R_{90}(\gamma')$ points towards the interior (except the end points). Otherwise it is negatively oriented.

Let $\gamma : [a,b] \to \mathbb{R}^2$ be a piecewise regular plane curve with partition denoted by $a = t_0 < \dots < t_n = b$. Each non smooth point $\gamma(t_i)$ is called a corner of γ.

At this corner, there are two velocity vectors coming from left and right hand limits.

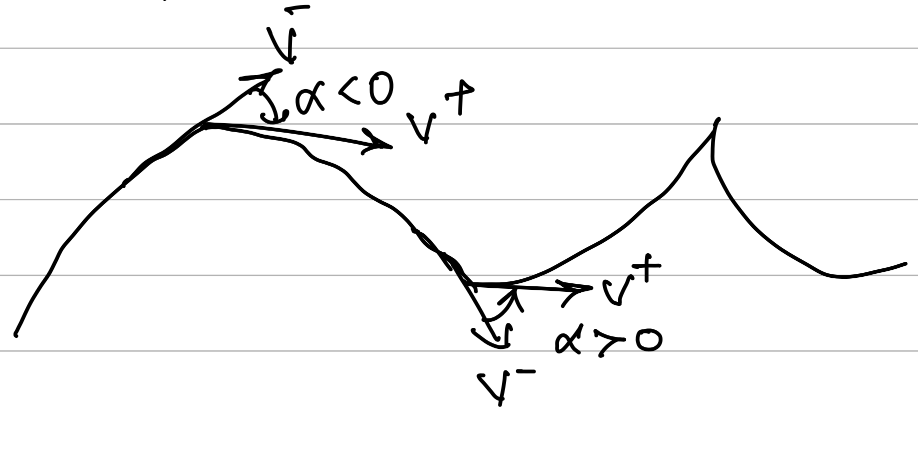
$$v^-(t_i) = \lim_{h \to 0^-} \frac{\gamma(t_i + h) - \gamma(t_i)}{h} = \lim_{t \to t_i^-} \gamma'(t)$$

$$v^+(t_i) = \lim_{h \to 0^+} \frac{\gamma(t_i + h) - \gamma(t_i)}{h} = \lim_{t \to t_i^+} \gamma'(t)$$

By regularity hypothesis, neither is zero.

The signed angle at $\gamma(t_i)$ denoted by $\alpha_i \in [-\pi, \pi]$ is defined such that its absolute value equals the smallest determination of the angle between $v^-(t_i)$ and $v^+(t_i)$.

The sign of α_i is defined to be positive if $v^+(t_i)$ is a counterclockwise rotation of $v^-(t_i)$ through this angle.



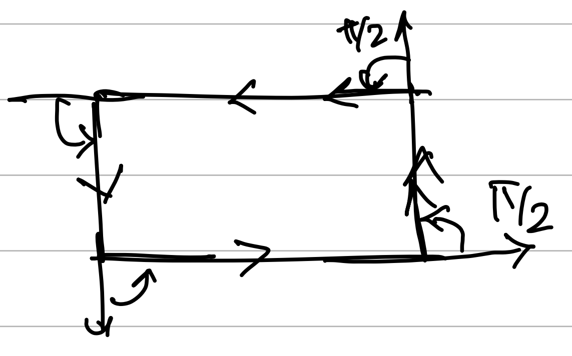
The corner $\gamma(t_i)$ is called a cusp if $v^+(t_i)$ is a negative scalar multiple of $v^-(t_i)$.

$\alpha_i = \pi$ if $v^-(t_i)$ points towards the exterior or $\alpha_i = -\pi$ if $v^-(t_i)$ points towards interior.

<u>Theorem</u> (Generalized Hopf's Umlaufsatz)

Let $\gamma : [a,b] \to \mathbb{R}^2$ be a unit speed positively oriented piecewise regular simple closed plane curve. Let κ_s denote its signed curvature function. Let $\{\alpha_i\}$ be the list of signed angles at its corners. Then

$$\int_a^b \kappa_s(t)\,dt + \sum_i \alpha_i = 2\pi$$



$$\boxed{2\pi = \sum \alpha_i}$$

If no corners, then $K_s = \theta'$

$$\int_a^b K_s \, dt = \theta(b) - \theta(a) = 2\pi (R_\gamma) = 2\pi$$

$$\Rightarrow R_\gamma = 1$$

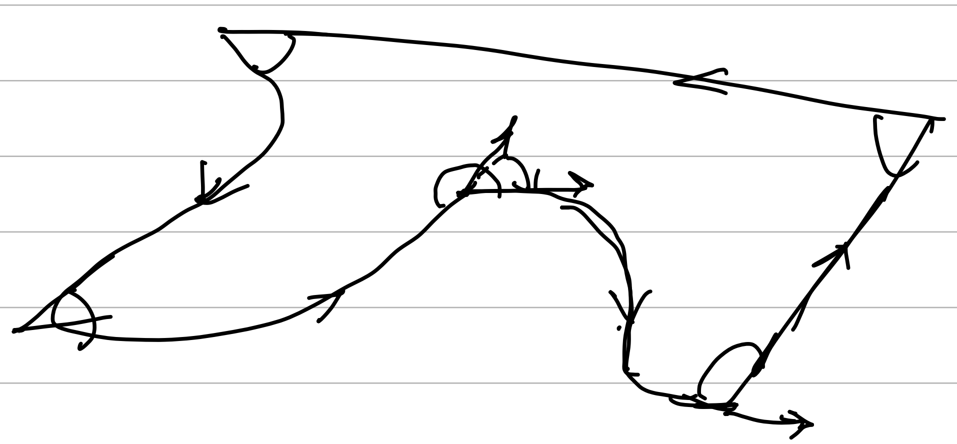
. Rephrase in terms of interior angles.

Let $\gamma : [a,b] \longrightarrow \mathbb{R}^2$ be a piecewise smooth simple closed plane curve with signed angles denoted by $\{\alpha_i\}$.

The i^{th} interior angle of γ, denoted by $\beta_i \in [0, 2\pi]$

Defined as

$$\beta_i = \begin{cases} \pi - \alpha_i & \text{if } \gamma \text{ is positively oriented} \\ \pi + \alpha_i & \text{if } \gamma \text{ is negatively oriented} \end{cases}$$



In this languqe, we have

$$\int_a^b K_s \, dt + \sum_i \alpha_i = 2\pi$$

$$\int_a^b K_s \, dt + \sum_i (\pi - \beta_i) = 2\pi$$

$$\int_a^b K_s \, dt + n\pi = 2\pi + \sum_i \beta_i$$

$$\int_a^b K_s \, dt = \sum_i \beta_i - (n-2)\pi$$

If smooth segments of γ are straight lines, then

$$\int_a^b K_s \, dt = 0$$

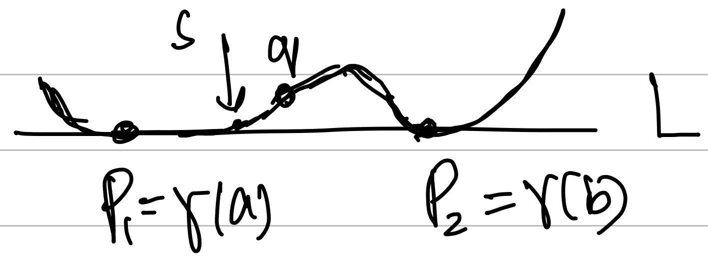
$$\Rightarrow \sum_i \beta_i = (n-2)\pi$$

Lemma Let C be the trace of a simple closed convex plane curve. and let L be a line

1) If L is tangent to C at two distinct points then C contains the entire segment of L between these two points.

Proof idea:

let P_1, P_2 s.t L is tangent to C.



let s denote last point of L past P_1

that is contained in C

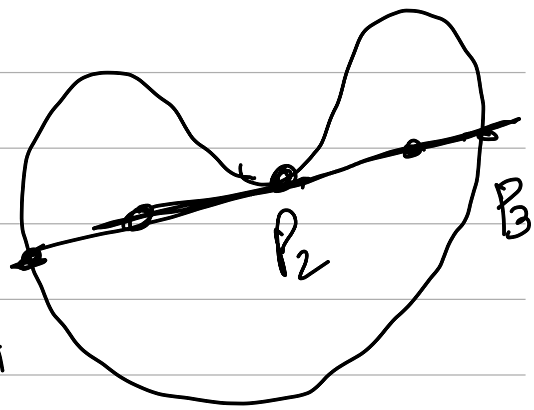
L is tangent to C at $\gamma(s)$ too.

For q immediately after s, tangent line at q places P_1, P_2 on opposite sides

$\Rightarrow\Leftarrow$

(2) If $C \cap L$ contains more than two points, then it contains the entire segment of L b/w any pair of these points.

Proof:- Suppose $L \cap C$ contains atleast 3 points order them $\{P_1, P_2, P_3\}$ so that P_2 is between P_1 and P_3 along L. Notice that tangent line to C at P_2 must be equal to L otherwise the other two points would lie on the opposite side of this tangent line. Since L is now a tangent line, convexity



$\Rightarrow$ C can't cross L at either of the other 2 points. So L must be tangent to C at all 3 points.

Follows from (1)

Prop:- Let γ be a simple closed plane curve. Let C denote its trace. Let I be its interior. TFAE

1) γ is convex: C lies on one side of each of its tangent lines.

2) The line segment joining any two points of I lies entirely in I.

3) κ_s does not change sign.

Proof:- $1 \Rightarrow 2)$ By contradiction, Assume that

γ is convex yet there is a pair $p, q \in I$ (interior) such that line segment joining them does not lie in I. The infinite line containing p, q must intersect C in atleast three points.

By previous Lemma 2, C contains the corresponding segment of L. So, $p, q \in C$ contradicting they are interior points.

Try others.

Fenchel's Theorem

The total curvature of a regular curve $\gamma : [a,b] \to \mathbb{R}^n$ is defined as

$$\text{total curvature} = \int_a^b \kappa(t) \|\gamma'(t)\| \, dt$$

$$= \int_a^b \kappa(t) \, dt \quad \text{(for arc-length parametrization)}$$

Lemma:-

The total curvature of a simple closed plane curve is $\geq 2\pi$. ("$=$" iff it is convex).

Proof:-

Let γ be a unit speed simple closed curve.

By Hopf's Umlaufsatz, we have

$$\underset{\text{signed curvature}}{\int_a^b \kappa_s(t)\, dt} = \int_a^b \theta'(t)\, dt = \theta(b) - \theta(a)$$

$$= 2\pi R_\gamma = \pm 2\pi$$

$$\Rightarrow \int_a^b \kappa(t)\, dt = \int_a^b |\kappa_s(t)|\, dt \geq \left| \int_a^b \kappa_s(t)\, dt \right| = 2\pi$$

(= iff κ_s never changes sign (i.e. convex)).

<u>Frenchel's</u>

The total curvature of a closed curve in $\mathbb{R}^n$ is $\geq 2\pi$. with "=" iff it is a simple closed convex curve contained in a plane in $\mathbb{R}^n$.

The Isoperimetric Inequality.

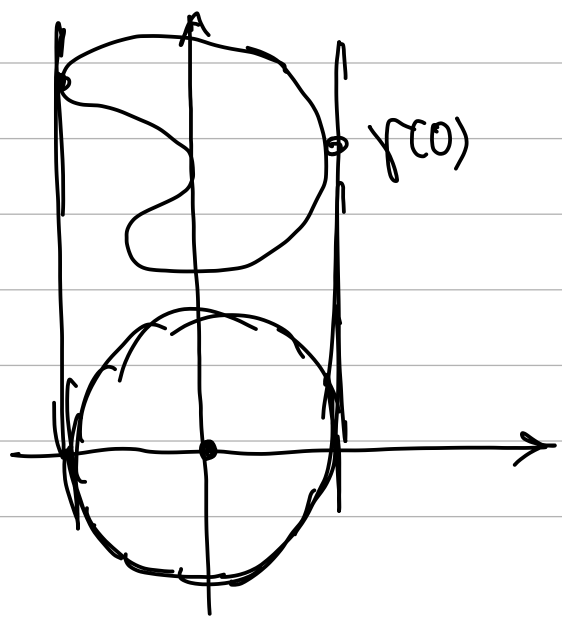
"Let C be a simple closed plane curve. If ℓ denotes the length of C and A denotes the area of the interior of C, then

$$\ell^2 \geq 4\pi A$$

with "=" iff C is a circle".

Proof:- Take two vertical lines that donot intersect C and move them together until they first touch C, so that C becomes tangent to both lines.

Let S^1 be a circle also tangent to both of these lines. It's radius r is half the distance between lines.



let $\gamma(t) = (x(t), y(t))$, $t \in [0, \ell]$ be arclength parametrization of γ. such that $\gamma(0)$ is the intersection of one C with one of the vertical lines. and $\gamma(t_0)$ is the intersection with other for some $t_0 \in (0, \ell)$.

We will choose a parametrization of $\beta : [0, \ell] \to \mathbb{R}^2$ s.t

x coordinates of β and γ agree.

This can be done by defining

$$\beta(t) = (x(t), \tilde{y}(t))$$

where $\tilde{y}(t) = \pm\sqrt{r^2 - (x(t))^2}$

with sign depending if $t \in [0, t_0]$ or $[t_0, \ell]$.

We know that A for C is

$$A = \int_0^\ell x y' \, dt$$

while Area for S^1 is

$$\pi r^2 = \int_0^{\ell} -\bar{y}x'\,dt$$

$F_1 = \langle 0, x \rangle$

$F_2 = \langle -y, 0 \rangle$

$$\Rightarrow A + \pi r^2 = \int_0^{\ell} (xy' - \bar{y}x')\,dt$$

$$\leq \int_0^{\ell} |xy' - \bar{y}x'|\,dt = \int_0^{\ell} \sqrt{(xy' - \bar{y}x')^2}\,dt$$

$$= \int_0^{\ell} \sqrt{x^2y'^2 - 2xx'y'\bar{y} + \bar{y}^2x'^2}\,dt$$

$$= \int_0^{\ell} \sqrt{(x^2+\bar{y}^2)(x'^2+y'^2) - (xx'+\bar{y}y')^2}\,dt$$

$$\leq \int_0^{\ell} \sqrt{(\ )(\ )}$$

$$\leq \int_0^{\ell} \sqrt{(x^2+\bar{y}^2)(1)}\,dt$$

$$= \int_0^{\ell} |\beta(t)|\,dt = r\ell = \ell r$$

$$\Rightarrow A + \pi r^2 \leq \ell r$$

Since

$$\sqrt{A}\sqrt{\pi r^2} \leq \frac{1}{2}(A + \pi r^2) \leq \frac{1}{2}\ell r$$

$$\Rightarrow \ell^2 \geq 4\pi A$$

"="

$$(x x' + \tilde{y} y')^2 = 0$$

$$\Rightarrow 0 = x x' + \tilde{y} y' = 0$$

$$0 = \langle x, \tilde{y} \rangle \cdot \langle x', y' \rangle = \langle \beta, \gamma' \rangle$$

Since $|\beta|$ is constant $\Rightarrow$

$$\langle \beta, \beta' \rangle = 0$$

$$\Rightarrow \gamma' \parallel \beta'$$

$$\gamma' = K\beta' \Rightarrow \langle x', y' \rangle = K \langle x', \tilde{y}' \rangle$$

$\Rightarrow K = 1$ (atleast when $x' + \tilde{y}' \neq 0$).

$$\Rightarrow \beta' = \gamma' \quad \forall t \in [0, \ell]$$

$$\Rightarrow \gamma = \beta + w, \quad w \in \mathbb{R}^2$$

$\boxtimes$