Isoperimetric Inequality

"let C be a simple closed plane curve. If "l" denotes the length of C and "A" denotes the area of the interior of C, then

$$l^2 \geq 4\pi A$$

with equality $\Leftrightarrow$ C is a circle".

For equality,

$$x x' + \tilde{y} y' = 0$$

$$0 = \langle x, \tilde{y} \rangle \cdot \langle x', y' \rangle$$

$$\Rightarrow 0 = \beta \cdot \gamma'$$

Also $|\beta| = r \Rightarrow \beta \cdot \beta' = 0$

$$\Rightarrow \beta' = K\gamma'$$

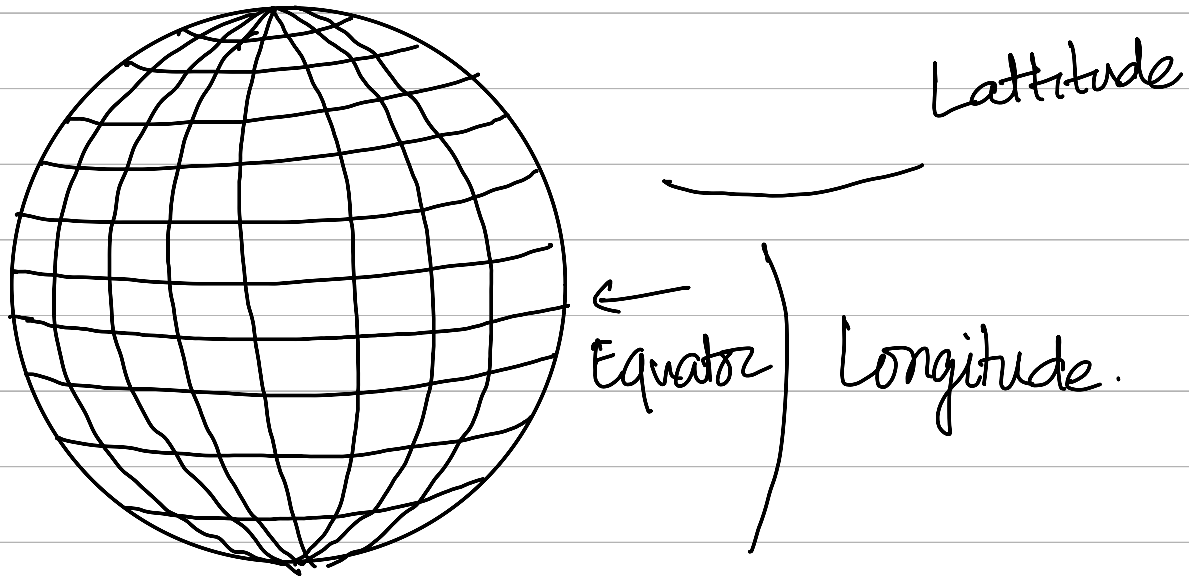
Since same x-coordinate $\Rightarrow$

$$\beta' = \gamma' \quad \forall\, t \in [0, l]$$

$\Rightarrow \gamma = \beta + w$ (Translation of the circle).

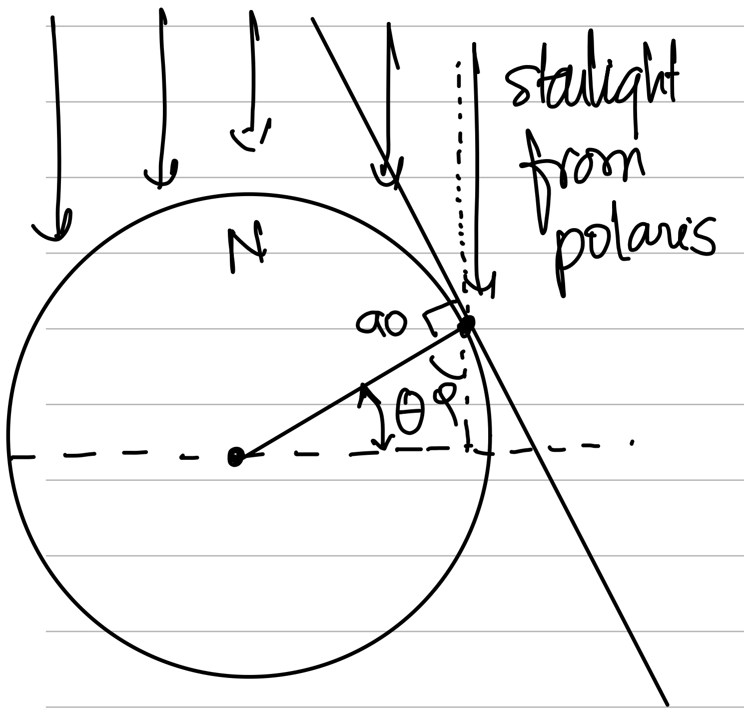
Huygen's Tautochrone clock

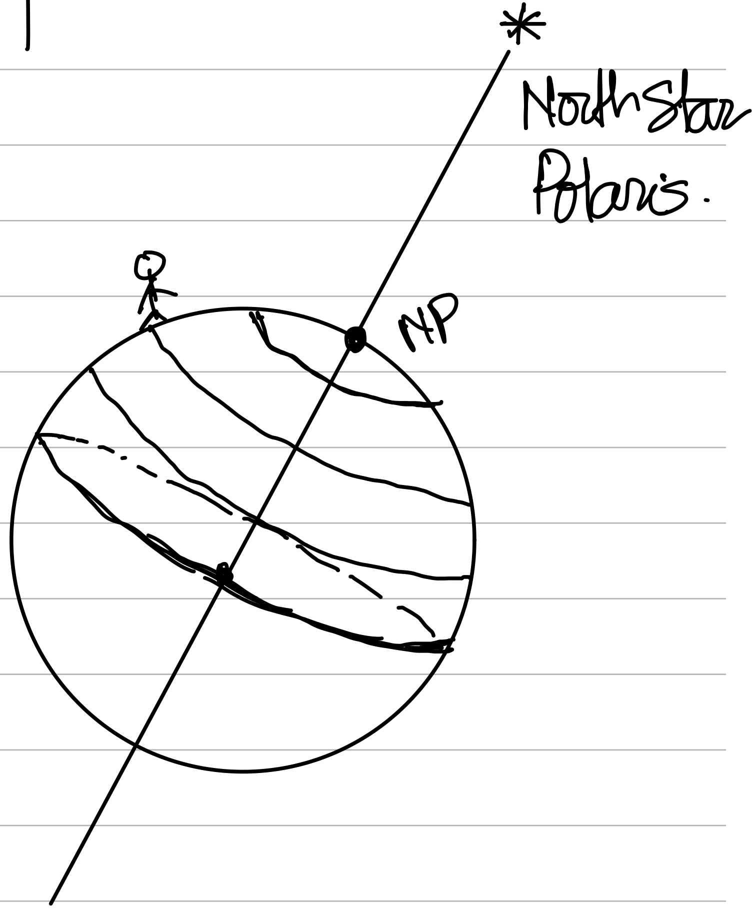
16-17 century Longitude problem.

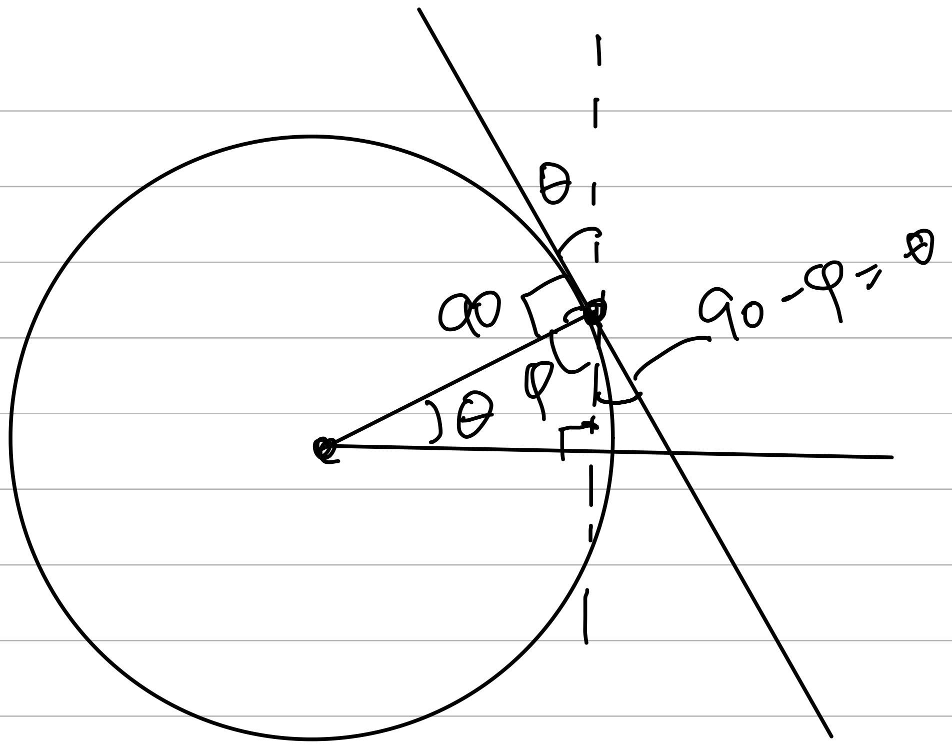


Lattitude was not a problem.

"Polaris North Star"







Angle of elevation to Polaris = Lattitude.

<u>Real problem was Longitude.</u>

As the lost life and lost gold piled up, several countries offered prizes and established scientific centers to solving the problem.

In 1714, British government established a "Board of Longitude" and offered a prize of 20,000 pounds.

Galileo, Newton and others came up with potential solutions based on astronomical observations.

Main alternative approach involved building a better clock.

Christiaan Huygens invented pendulum clocks but weren't best at sea.

Huygens was first to prove that circular arc is not isochronus.

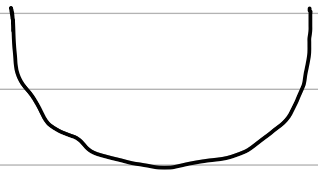
He was interested in the shape with the property that the time to reach the bottom is independent of object's starting position.

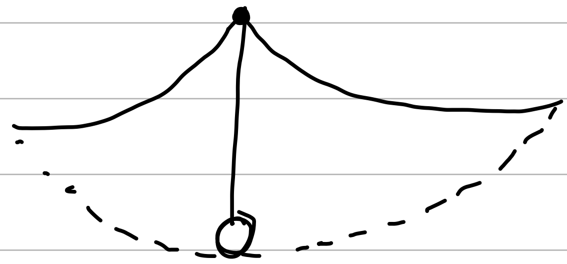
He called it a "tautochrone curve".

He designed bumpers against which the pendulum string would wrap.

Didn't work well.

Prize ultimately went to John Harrison (springs & balances).



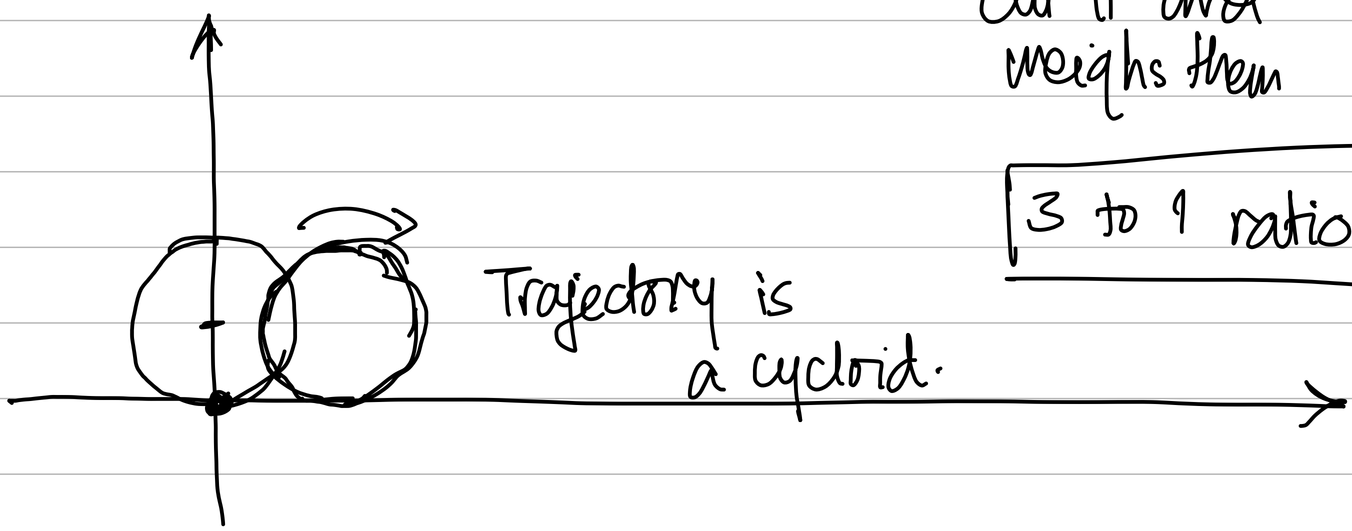


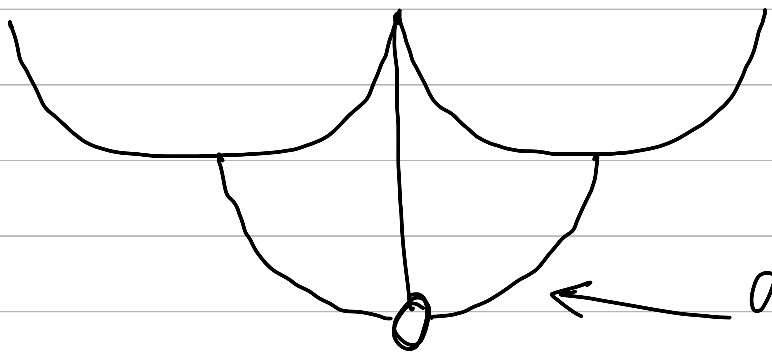
Even though Huygen's failed, his mathematics lead to many important discoveries.

Start with cycloid:

Galliteo
cut it and
weighs them

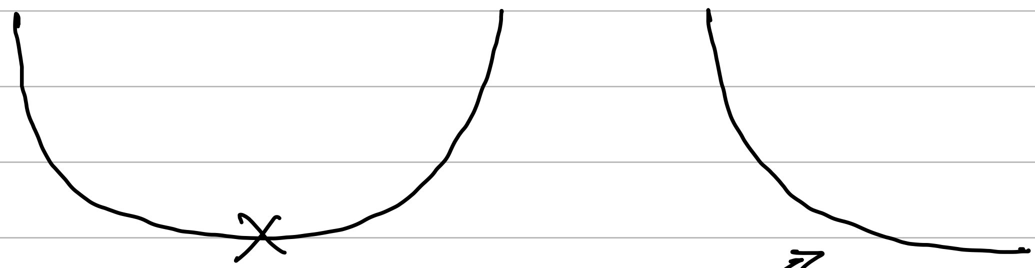
$\boxed{\text{3 to 1 ratio}}$





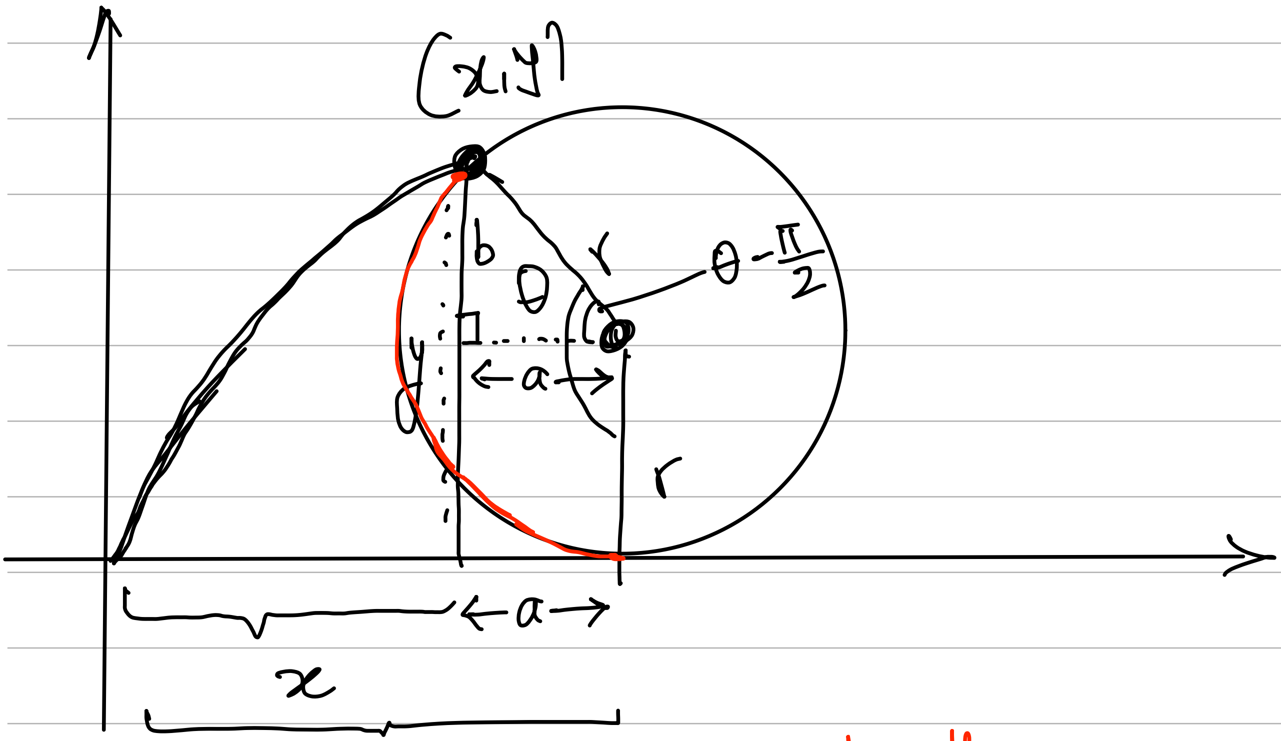
also a cycloid

Split a cycloid in half



Brachistochrone curve

Equation of a cycloid



$x + a$ = red arclength $= r\theta$

$x = r\theta - a$

Also $y = r + b$

$$\sin\left(\theta - \frac{\pi}{2}\right) = \frac{b}{r} \Rightarrow b = -r\cos\theta$$

$$\cos\left(\theta - \frac{\pi}{2}\right) = \frac{a}{r} \Rightarrow a = r\sin\theta$$

$$\Rightarrow x = r\theta - r\sin\theta$$
$$y = r - r\cos\theta$$

: For a unit circle:

$r(t) = \langle t - \sin t, 1 - \cos t \rangle$

Inverted cycloid would be

$$\tilde{r}(t) = \langle t - \sin t, \cos t - 1 \rangle$$

<u>Thm:</u> The inverted cycloid is a Tautochrone curve.

Proof:- Consider an object beginning at $\tilde{r}(t_0) = (x_0, y_0)$ and sliding under the influence only of gravity to the bottom of the curve $\tilde{r}(\pi)$.

Need to show that the time required is independent of t_0.

It suffices to assume $t_0 \in [0, \pi)$ (because of symmetry).

Let's relate the following variables

t, x, y, s, T.

$$s(t) = \int_{t_0}^{t} \|\bar{\gamma}'(u)\| \, du$$

T is the time parameter. i.e. $T(t)$ is the time required for a particle to slide from position $\bar{\gamma}(t_0)$ to $\bar{\gamma}(t)$.

Note that

$$\left(\frac{ds}{dt}\right)^2 = \left|\frac{d\gamma'}{dt}\right|^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 =$$

$$(1-\cos t)^2 + \sin^2 t = 2 - 2\cos t$$

By Energy conservation:

$$\frac{1}{2}mv^2 = mg(y_0 - y)$$

where v is the object's speed, measured as

$$\frac{ds}{dT} = v = \sqrt{2g(y_0 - y)}$$

Also by Chain Rule:

$$\frac{ds}{dT} = \frac{ds}{dT} \times \frac{dT}{dt} \Rightarrow \frac{dT}{dt} = \frac{ds/dt}{ds/dT} = \frac{\sqrt{2-2\cos t}}{\sqrt{2g(y_0 - y)}}$$

$$\frac{dT}{dt} = \frac{\sqrt{1-\cos t}}{\sqrt{g}\sqrt{\cos t_0 - \cos t}}$$

$$T = \int_{t_0}^{\pi} \frac{dT}{dt}\,dt = \frac{1}{\sqrt{g}}\int_{t_0}^{\pi} \sqrt{\frac{1-\cos t}{\cos t_0 - \cos t}}\,dt$$

$$\sin^2 t = \frac{1-\cos(2t)}{2} \Rightarrow 1-\cos t = 2\sin^2(t/2)$$

$$T = \frac{1}{\sqrt{g}} \int_{t_0}^{\pi} \frac{\sqrt{2}\sin(t/2)}{\sqrt{\cos(t_0) - \cos t}}\, dt$$

Also $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$

$$\Rightarrow \cos\theta = 2\cos^2\left(\frac{\theta}{2}\right) - 1$$

$$T = \frac{1}{\sqrt{g}} \int_{t_0}^{\pi} \frac{\sin(t/2)}{\sqrt{\cos^2(t_0/2) - \cos^2(t/2)}}\, dt$$

$$u = \cos(t/2) \Rightarrow du = -\frac{1}{2}\sin\left(\frac{t}{2}\right) dt$$

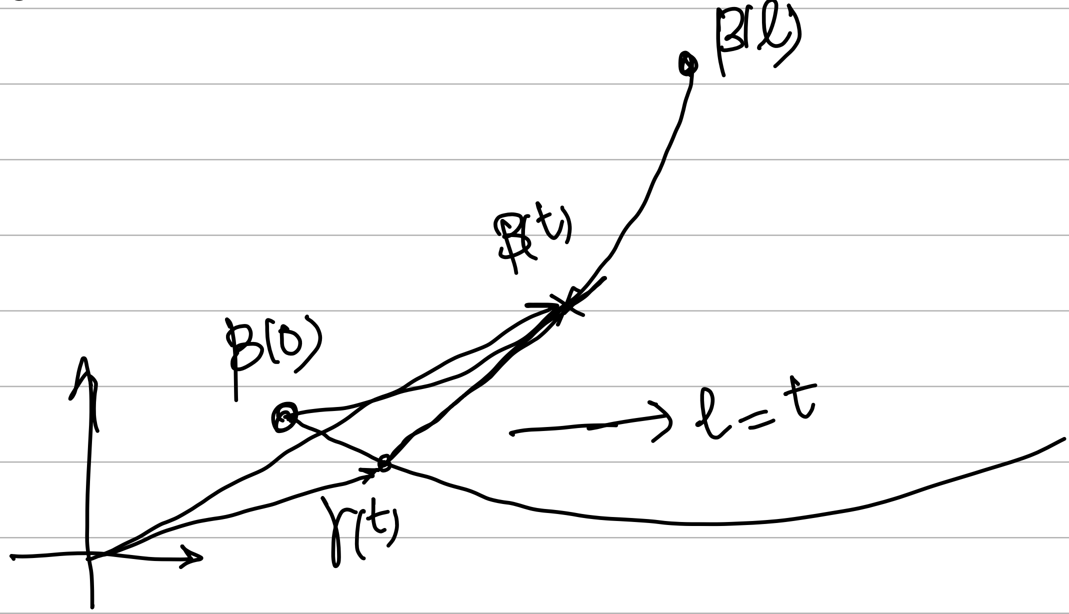
$$T = -\frac{1}{\sqrt{g}} \int_{\cos\left(\frac{t_0}{2}\right)}^{0} \frac{2\, du}{\sqrt{\cos^2\left(\frac{t_0}{2}\right) - u^2}}$$

$$= \frac{-2}{\sqrt{g}} \sin^{-1}\left(\frac{u}{\cos\left(\frac{t_0}{2}\right)}\right)\Bigg|_{\cos\left(\frac{t_0}{2}\right)}^{0} \qquad = \frac{\pi}{\sqrt{g}}.$$

Huygen's second problem was to determine the shape of the bumpers.

For this let $\beta : [0, \ell] \to \mathbb{R}^2$ be a parametrized plane curve (parametrized by arclength) (bumper curve)

and let $\gamma : [0, \ell] \to \mathbb{R}^2$ be the induced parametrization of the path that bob will follow.



Easy to see that $\beta(0) = \gamma(0)$

$$\gamma(t) + (\beta'(t)\, t) = \beta(t) \Rightarrow \gamma(t) = \beta(t) - t\beta'(t)$$

(1)

Def:- Let B be an oriented plane curve with non-zero curvature.

An involute of B is a plane curve G that has parametrization $\gamma: (0,l) \to \mathbb{R}^2$ of the form

$$\gamma(t) = \beta(t) - (t+\lambda_0)\beta'(t)$$

where $\beta: (0,l) \to \mathbb{R}^2$ is an arc-length parametrization of B and $\lambda_0 \in \mathbb{R}$ is a constant with $\lambda_0 \in (-l, 0)$.

Note that

$$\gamma' = \beta' - \beta' - (t+\lambda_0)\beta'' \neq 0$$

$\Rightarrow$ γ is regular.

If β is not parametrized by its arc-length then we can choose

$$\gamma(t) = \beta(t) - (s(t)+\lambda_0)\frac{\beta'(t)}{\|\beta'(t)\|}$$

Huygens wanted to solve the inverse problem:

Find the bumper curve B whose involute is the tautochrone curve G.

Def ②:-

let G be an oriented plane curve

parametrized as $\gamma : (0,l) \longrightarrow \mathbb{R}^2$

Assume that for all $t \in (0,l)$, $\kappa(t) \neq 0$, $\kappa'(t) \neq 0$
let $\vec{n}$ be the unit normal to γ.
The evolute B, of G is the plane curve with parametrization $\beta : (0,l) \longrightarrow \mathbb{R}^2$ given by

$$\beta(t) = \gamma(t) + \frac{1}{\kappa(t)} n(t)$$

Is β regular?

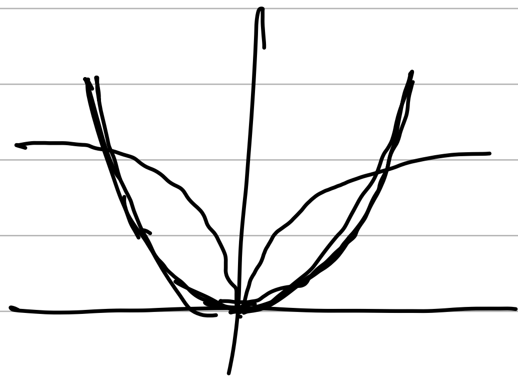
Prop:-

i) With assumptions of ①, B is the evolute of G.

ii) " " ②, G is an involute of B

Solution of Huygens's problem?

Q:- Prove that the evolute of $y=x^2$ is
$y=\frac{1}{2}+3\left|\frac{x}{4}\right|^{2/3}$.



$r(t)=\langle t, t^2\rangle$

$r'=\langle 1,2t\rangle,\ r''=\langle 0,2\rangle$

$$r'\times r''=\begin{vmatrix} i & j & k \\ 1 & 2t & 0 \\ 0 & 2 & 0 \end{vmatrix}=2$$

$$K=\frac{2}{(1+4t^2)^{3/2}}$$

$$t=\left\langle \frac{1}{(1+4t^2)^{1/2}},\ \frac{2t}{(1+4t^2)^{1/2}}\right\rangle$$

$$t' = \left\langle -\frac{1}{2}(1+4t^2)^{-3/2}(8t),\ 2(1+4t^2)^{-3/2} \right\rangle$$

$$\|t'\| = \sqrt{\frac{16t^2}{(1+4t^2)^3} + \frac{4}{(1+4t^2)^3}} = \frac{2\sqrt{1+4t^2}}{(1+4t^2)^3}$$

$$n = \left\langle -\frac{2t}{\sqrt{1+4t^2}},\ \frac{1}{\sqrt{1+4t^2}} \right\rangle$$

$$r = \langle t, t^2 \rangle + \left(\frac{(1+4t^2)^{3/2}}{2}\right) \times \frac{1}{\sqrt{1+4t^2}} \langle -2t, 1 \rangle$$

$$= \left\langle -4t^3,\ \frac{1}{2} + 3t^3 \right\rangle.$$