Euclidian spaces

$(\mathbb{R}^n, +, \cdot)$ vector-space. (metric structure).

Parametrized Curves:

A parametrized curve in $\mathbb{R}^n$ is a smooth function $\gamma : I \to \mathbb{R}^n$.

<u>Example</u>
- $\gamma(t) = (\cos t, \sin t)$
- $\gamma(t) = (t, f(t))$ for any smooth function f.

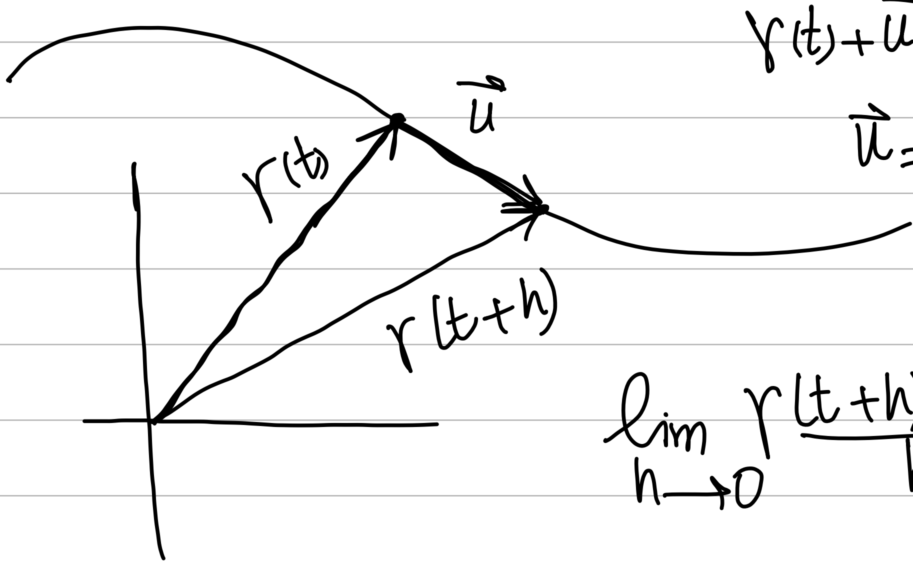
  parametrization of a graph $y = f(x)$.
- $\gamma(t) = (t, t^{1/3})$ not smooth.

Def:- If $\gamma : I \to \mathbb{R}^n$ given by $\gamma(t) = (x_1(t), \ldots, x_n(t))$

then $\gamma' = (x_1'(t), \ldots, x_n'(t))$.

Prop:- The derivative of $\gamma : I \to \mathbb{R}^n$ at time $t \in I$ is given by

$$\gamma'(t) = \lim_{h \to 0} \frac{\gamma(t+h) - \gamma(t)}{h}$$



$\gamma(t) + \vec{u} = \gamma(t+h)$

$\vec{u} = \gamma(t+h) - \gamma(t)$

$$\lim_{h \to 0} \frac{\gamma(t+h) - \gamma(t)}{h}$$

tangent (velocity vector)
(direction of motion)

$\|\gamma'(t)\|$ speed of object at time "t".

$$S = vt \Rightarrow s = \int_{t_0}^{t_1} \|\gamma'(t)\| dt$$

$\|\cdot\|$ is the norm of a vector in $\mathbb{R}^n$.

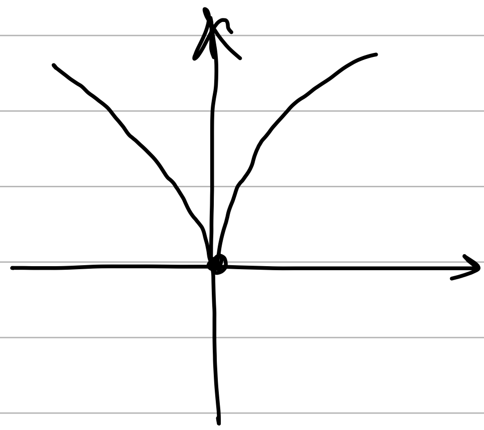
Def:- (Regular curve)

$\gamma : I \to \mathbb{R}^n$ is called regular if its speed is always non-zero i.e $\|\gamma'(t)\| \neq 0 \ \forall t \in I$.

It is called <u>unit speed</u> or <u>parametrized by arc-length</u> if $\|\gamma'(t)\| = 1 \ \forall t \in I$.

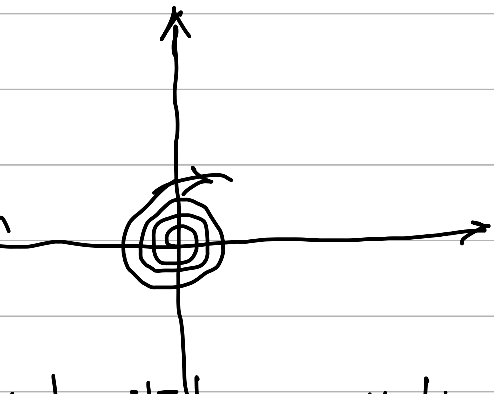
<u>Example</u> $\gamma(t) = (t^3, t^2)$

Smooth but not regular



<u>Exercise</u>: $\gamma(t) = c(e^{\lambda t}\cos t, e^{\lambda t}\sin t)$

Logarithmic spiral



Show that it has finite length for $\lambda < 0$.

Eventhough it wraps origin infinitely many times?

$$S = \int_0^{\infty} \sqrt{5}e^{-2t}\,dt = \frac{\sqrt{5}}{2}$$

$$\boxed{r(t+2\pi) = c\left(e^{\lambda t}e^{2\pi}\cos t,\ e^{\lambda t}e^{2\pi}\sin t\right)}$$

Exercise: Find the max/min Speed of

$$r(t) = (a\cos t, b\sin t)$$

$$\|r'(t)\| = \sqrt{a^2\sin^2 t + b^2\cos^2 t}$$

$$F(t) = a^2\sin^2 t + b^2\cos^2 t$$

$$F'(t) = 0 \Rightarrow (a^2-b^2)\sin(2t) = 0 \Rightarrow t = k\pi \text{ or } \frac{\pi}{2} + k\pi$$

Inner product on $\mathbb{R}^n$

- $\langle x, x\rangle \geq 0$, $\langle x, x\rangle = 0 \Leftrightarrow x = 0$
  $= \|x\|^2$
- $\langle x, y\rangle = \langle y, x\rangle$
- $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$
  $\langle x, \gamma y + \delta z\rangle = \_\_ + \_\_$ (Bilinear)

- Schwarz Inequality

$$\langle x,y\rangle \le |\langle x,y\rangle| \le \|x\|\|y\|$$

$$\langle x,x\rangle = \|x\|^2$$

$$\|x\| = \sqrt{\langle x,x\rangle}$$

$X=\{0,1\}$

$$d(x,y)=\begin{cases}0 & x=y\\ 1 & x\neq y\end{cases}$$

$$d(x,y)=\|x-y\|^2$$

Suppose X is a linear space with $\|\cdot\|$

$\|1\| = d(0,1) = 1$

$\|2\| = 2\|1\| = 2 = d(0,2)$ (X).

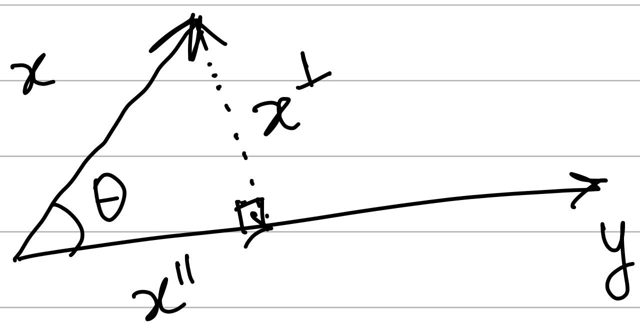
$$\langle x,y\rangle = \|x\|\|y\|\cos\theta$$

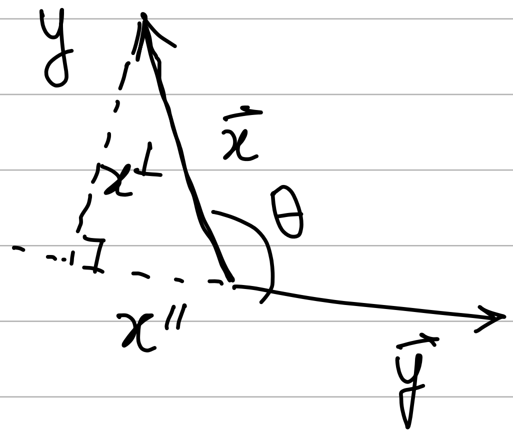
x,y orthogonal if $\langle x,y\rangle = 0$.

Prop: If $x, y \in \mathbb{R}^n$ with $\|y\| \neq 0$, then there is a unique way to write x as a sum of two vectors

$$x = x^{\parallel} + x^{\perp}$$

$x^{\parallel}$ is parallel to y and $x^{\perp}$ is orthogonal to y.





Proof: Enough to show that there is a unique $\lambda \in \mathbb{R}$ s.t

$$x^{\parallel} = \lambda \frac{y}{\|y\|}, \quad \text{then } x = x^{\parallel} + x^{\perp}$$

$x^{\perp}$ is orthogonal to $y \Rightarrow$

$$0 = \langle x^{\perp}, y \rangle = \langle x - x^{\parallel}, y \rangle = \langle x, y \rangle - \lambda \frac{\langle y, y \rangle}{\|y\|}$$

$$\Rightarrow \lambda = \frac{\langle x, y \rangle}{\|y\|} \quad \blacksquare$$

$x^{\parallel}$ is called projection of x on y.

- A set $Y = \{y_1, \dots, y_n\}$ is called orthonormal if

$$\langle y_i, y_j \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Ex! Show that every orthonormal set is linearly independent.

Prop: If $V \subset \mathbb{R}^n$ is a subspace and $x \in \mathbb{R}^n$ then there is a unique way to write x as

$$x = x^{\parallel} + x^{\perp}$$

$x^{\parallel} \in V$ and $x^{\perp} \in V^{\perp}$.

Proof (Idea)

let $Y = \{y_1, \dots, y_k\}$ be an orthonormal basis of V. Show that $\exists!$ choice of scalars $a_1, \dots, a_k \in \mathbb{R}$

$$x^{\parallel} = a_1 y_1 + \dots + a_k y_k \text{ and } x^{\perp} = x - x^{\parallel}$$

then

$$\langle x^{\perp}, y_i \rangle = 0 \quad \text{solve for } a_i.$$

Standard orthonormal basis of $\mathbb{R}^n$.

$$e_i = (0, \dots, 1_i, \dots, 0).$$

Lemma:

If $\gamma, \beta : I \to \mathbb{R}^n$ is a pair of curves and $c : I \to \mathbb{R}$ is a smooth function then

i) $\frac{d}{dt}\langle \gamma, \beta \rangle = \langle \gamma', \beta \rangle + \langle \gamma, \beta' \rangle$

ii) $\frac{d}{dt}(c\gamma) = c'\gamma + c\gamma'$.

Proof (left as an exercise).

Prop:- let $\gamma, \beta : I \to \mathbb{R}^n$ be a pair of curves.

1) If γ has a constant norm, then γ' is orthogonal to γ for all $t \in I$.

2) If $\gamma(t)$ is orthogonal to $\beta(t)$ for all $t \in I$ then
$$\langle \gamma', \beta \rangle = -\langle \gamma, \beta' \rangle$$

Proof :- (Trivial)

Prop: If $\gamma : I \to \mathbb{R}^n$ is a curve with constant speed, then γ' is orthogonal to γ''. ($\forall t \in I$)

Example :-

Let $\gamma : [a,b] \longrightarrow \mathbb{R}^n$ be a curve.

Define $p = \gamma(a)$ and $q = \gamma(b)$ and let L be the arclength of γ.

Prove that $L \geq \underset{\substack{\downarrow \\ \text{straight line}}}{\|q-p\|}$

Sol:- $L = \int_a^b \|\gamma'(t)\|\,dt \geq \int_a^b \langle \gamma', n \rangle\,dt$

Consider $n = \dfrac{q-p}{\|q-p\|}$

$$= \int_a^b \frac{d}{dt} \langle \gamma, n \rangle\,dt$$

$$= \langle q-p, n \rangle = \|q-p\|$$

- $a(t) = r''(t)$ (acceleration vector)

Prop:- $$\frac{\langle a(t), v(t)\rangle}{\|v(t)\|} = \frac{d}{dt}\|v(t)\|$$

Proof:- $$\frac{d}{dt}\langle v(t), v(t)\rangle^{1/2} = \frac{1}{2}\langle -,-\rangle^{-1/2} 2\langle v, v'\rangle$$

$$= \frac{\langle a, v\rangle}{\|v\|}$$

Exercise:: If r is a curve in $\mathbb{R}^n$ with $|r(t)| = c$, prove that $\langle a, -r(t)\rangle = \|v\|^2$

$$\langle r, r\rangle = c^2$$

$$2\langle r, r'\rangle = 0 \Rightarrow \langle r', r'\rangle + \langle r, r''\rangle = 0$$