Reparametrization

$\gamma : I \to \mathbb{R}^n$ (regular curve) (Equivalence relation?)

$\gamma(I) = \{ \gamma(t) : t \in I \}$ is called trace of γ.

Example $\gamma_1(t) = (t, t^2)$, $t \in [-2,2]$

$\gamma_2(t) = (2t, (2t)^2)$ $t \in [-1,1]$

$\varphi : [-1,1] \to [-2,2]$, $\varphi(t) = 2t$

s.t

$\gamma_2(t) = (\gamma_1 \circ \varphi)(t)$

Def: Suppose that $\gamma : I \to \mathbb{R}^n$ is a regular curve. A reparametrization of γ is a function of the form $\tilde{\gamma} = \gamma \circ \varphi : \tilde{I} \to \mathbb{R}^n$, where $\tilde{I}$ is an interval and $\varphi : \tilde{I} \to I$ is a smooth bijection with nowhere vanishing derivative.

$$\tilde{\gamma}'(t) = \gamma'(\varphi(t))\boxed{\varphi'(t)}$$

$\neq 0$ to preserve regularity

regularity & smoothness of $\gamma \circ \varphi \Rightarrow \varphi' > 0$ or $\varphi' < 0$ on entire I.

Defi:-

- $\tilde{\gamma}$ is called orientation preserving if $\varphi' > 0$

and orientation reversing if $\varphi' < 0$.

[$\gamma \sim \gamma'$ if φ is orientation preserving.] [Prove $\sim$ is equivalence]

Prop:- A regular curve $\gamma : I \to \mathbb{R}^n$ can be reparametrized by arc-length. (i.e $\exists$ a unit speed reparametrization of γ).

Proof:- Choose any $t_0 \in I$ and consider

$$s(t) = \int_{t_0}^{t} \|\gamma'(u)\| du$$

* If f is continuous strictly monotonic funcn on I, then its

inverse function $g = f^{-1}$ is defined on $J = f(I)$. [Bartle 3rd ed p164]

Let $\tilde{I} = s(I)$. By fundamental theorem of

Calculus, $s'(t) = \|\gamma'(t)\| \neq 0$ (use of regularity)

$\Rightarrow$ s is a smooth bijection from I to $\tilde{I}$.
(nowhere vanishing derivative).
$\Rightarrow$ s has an inverse.

Inverse of a smooth bijective map always smooth?

$$\varphi : \tilde{I} \to I$$

φ is also smooth with nowhere vanishing derivative.

$$\tilde{\gamma} = \gamma \circ \varphi$$

$$\|(\tilde{\gamma})'\| = \|\gamma'(\varphi(t))\, \varphi'(t)\| = \frac{\|\gamma'(\varphi(t))\|}{\|\gamma'(\varphi(t))\|} = 1$$

$1 = (s \circ \varphi)'(t) = s'(\varphi(t))\varphi'(t)$

<u>Inverse function Thm</u> $F : U \subset \mathbb{R}^n$ be C^1. Suppose $p \in U$ and $F'(p)$ is invertible i.e $|J_F(p)| \neq 0$, then $\exists$ $V, W \subset \mathbb{R}^n$ s.t $p \in V \subset U$, $F(V) = W$ and $F|_V$ is one-to-one.

Q1. A function that is everywhere locally invertible but not globally.

$$f(x) = \begin{cases} x^2 ; & x \geq 0 \\ x^2 - 1 ; & x < 0 \end{cases}$$

For $x > 0$, $y \to \sqrt{y}$ for $x < 0$ $y \to -\sqrt{y+1}$

$$y \longmapsto \begin{cases} -\sqrt{y+1} & , y \in (-1, 0) \\ \sqrt{y} & , y \in [0, 1) \end{cases}$$

not injective $\Rightarrow$ no global inverse.

Ex! $F : \mathbb{R}^2 \to \mathbb{R}$

$$F(x, y) = \begin{bmatrix} e^x \cos y \\ e^x \sin y \end{bmatrix}$$

$$J = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = e^{2x} \neq 0$$

$$F(x, y) = F(x, y + 2\pi)$$

Exercise:

Verify that

$$\tilde{\gamma}(t) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right), \ t \in \mathbb{R}$$

is a reparametrization of $\gamma(s) = (\cos s, \sin s)$.

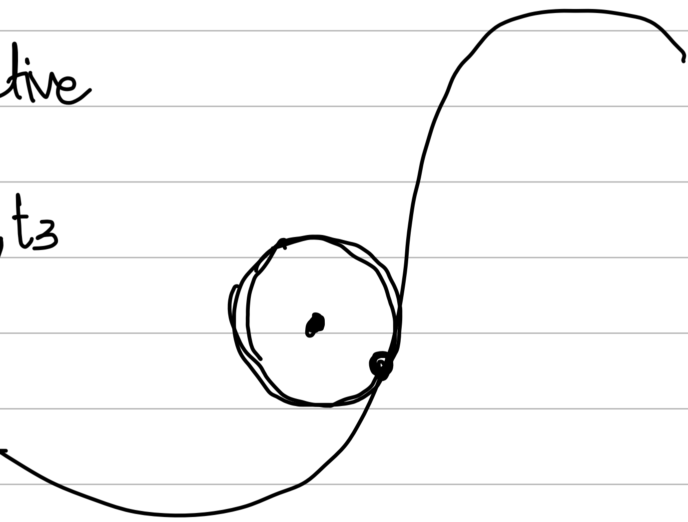
$\left[\text{Hint: } \quad t = \tan\left(\frac{s}{2}\right)\right]$

- Find arclength parametrization of $\gamma(t) = (a\cos(\alpha t), a\sin(\alpha t), bt)$

Curvature:

γ is locally injective

we can find t_1, t_2, t_3 near t,



$c(t_i)$ are distinct

Assume they don't lie on a line

we have a unique circle passing through these three points.

let its center be $C(t_1, t_2, t_3)$.

Consider a function

$$t \longmapsto \langle \gamma(t) - C(t_1,t_2,t_3), \gamma(t) - C(t_1,t_2,t_3) \rangle$$
$$= \|\gamma(t) - C(t_1,t_2,t_3)\|^2$$

$f(t) = \|\gamma(t) - C(t_1,t_2,t_3)\|^2$

Take same value at t_1, t_2, t_3

$f(t_1) = f(t_2) = f(t_3)$

$\Rightarrow \exists\, \xi_1 \in (t_1,t_2)$ & $\xi_2 \in (t_2,t_3)$

such that

$$f'(\xi_1) = 0, \quad f'(\xi_2) = 0$$

$$\Rightarrow 0 = \langle \gamma'(\xi_i), \gamma(\xi_i) - C(t_1,t_2,t_3) \rangle$$

Likewise

$$t \longmapsto \langle \gamma'(t), \gamma(t) - C(t_1,t_2,t_3) \rangle$$

has same value at ξ_1 & ξ_2.

$\Rightarrow \exists\, \eta \in (\xi_1, \xi_2)$ s.t

$$\langle \gamma''(\eta), \gamma(\eta) - C(t_1, t_2, t_3) \rangle = -\langle \gamma'(\eta), \gamma'(\eta) \rangle$$

[If!] limiting circle exists, we must have

$(t_1, t_2, t_3 \longrightarrow t)$

$C(t_1, t_2, t_3) \longrightarrow C$

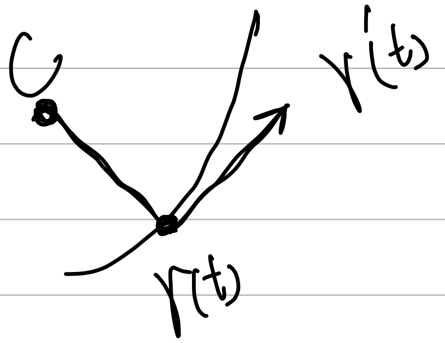
i') $\langle \gamma'(t), \gamma(t) - C \rangle = 0$

ii) $\langle \gamma''(t), \gamma(t) - C \rangle = -\langle \gamma'(t), \gamma'(t) \rangle$

i) show circle through $\gamma(t)$ with center C must be tangent to γ at $\gamma(t)$.

If $\gamma'' = K\gamma'$

then



$\langle K\gamma', \gamma - C \rangle = K\langle \gamma, \gamma - C \rangle = 0$ (a contradiction) so $\gamma'' \neq k\gamma'$

Consider WLOG arclength parametrization

$$\|\gamma'(s)\| = 1$$

$$\Rightarrow \langle \gamma'(s), \gamma'(s) \rangle = 1$$

$$\Rightarrow \langle \gamma'', \gamma' \rangle = 0 \quad \text{i.e } \gamma'' \perp \gamma'$$

we must have

$$a\gamma''(s) = \gamma(s) - c \quad \text{———} \quad ✪$$

by (ii) we have

$$\langle \gamma''(s), a\gamma''(s) \rangle = -1$$

$$\Rightarrow a = \frac{-1}{\|\gamma''(s)\|^2}$$

$$✪ \Rightarrow \|\gamma(s) - c\| = |a| \, \|\gamma''(s)\| = \boxed{\frac{1}{\|\gamma''(s)\|}}$$

$$\kappa = \frac{1}{R} = \|\gamma''(s)\|$$

Theorem

Let $c : [a,b] \to \mathbb{R}^2$ be a C^2-curve parametrized by arc-length. If $c''(s) \neq 0$, then $c(s_1), c(s_2), c(s_3)$ donot lie on a straight line. As $s_i \to s$, the unique circle through $c(s_i)$ approaches a circle passing through $c(s)$ whose radius is $\frac{1}{\|c''(s)\|}$ and

whose center lies on the line through $c(s)$ perpendicular to the tangent line through $c(s)$.

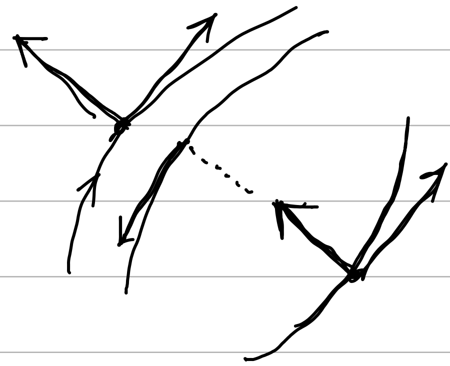
If $c''(s) = 0$, evenif $c(s_i)$ donot lie on a line, the circles through them donot approach a limiting circle.

curvature $\|\gamma''(s)\|$

$\Rightarrow \kappa(s) > 0$

$t(s) = \gamma'(s)$

$n(s) = \langle -t^2(s), t^1(s) \rangle$

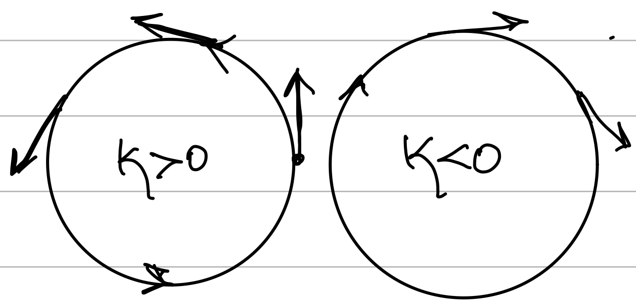


Since $\left\|\frac{dt}{ds}\right\| = \kappa$

$\downarrow$

Defines

$t'(s) = \kappa(s)\, n(s)$



$\gamma(s) = (\gamma_1(s), \gamma_2(s))$

$\langle \gamma', \gamma' \rangle = 1 \Rightarrow 2\langle \gamma', \gamma'' \rangle = 0 \Rightarrow \gamma' \perp \gamma''$.

linearly independent

$\|\gamma''\| = \det(\gamma', \gamma'') = \kappa(s)$

$= \left| \begin{pmatrix} \gamma_1' & \gamma_1'' \\ \gamma_2' & \gamma_2'' \end{pmatrix} \right|$

Non-arc length parametrization

$C : [a, b] \to \mathbb{R}^n$

$S : [a, b] \to [0, L]$

Define $\gamma = C \circ S^{-1} : [0, L] \to \mathbb{R}^n$

we have $\quad C = \gamma \circ S$

$$\frac{dc}{dt} = \gamma'(s) \frac{ds}{dt}$$

$$\frac{d^2c}{dt^2} = \gamma''(s) \left(\frac{ds}{dt}\right)^2 + \gamma'(s) \frac{d^2s}{dt^2}$$

$$\Rightarrow \gamma'(s) = \frac{dc/dt}{ds/dt}$$

and $$\gamma''(s) = \frac{d^2c/dt^2 - d^2s/dt^2 \left(\frac{dc/dt}{ds/dt}\right)}{\left(\frac{ds}{dt}\right)^2}$$

$$= \left(\frac{d^2c}{dt^2} - \alpha \frac{dc}{dt}\right) \Big/ (ds/dt)^2$$

Now $\quad \kappa(s) = \det\left(\gamma'(s), \gamma''(s)\right)$

$$= \frac{1}{\left(\frac{ds}{dt}\right)^3} \det\left(\dot{c}, \ddot{c} - \alpha\dot{c}\right)$$

$$= \frac{\det(\dot{c}, \ddot{c})}{\left(\frac{ds}{dt}\right)^3}$$

also $\quad \frac{ds}{dt} = \left\| \frac{dc}{dt} \right\| = \sqrt{\dot{c}_1^2 + \dot{c}_2^2}$

$$\langle \gamma', \gamma' \rangle = 1 \Rightarrow \langle \gamma', \gamma'' \rangle = 0$$

$\Rightarrow \gamma' \,\&\, \gamma''$ always orthogonal

$\Rightarrow \gamma', \gamma''$ always lin. independent.