"Planar case"

$\kappa = 0$

$c''(s) = 0 \Rightarrow c(s) = as + b$ straight line

$\kappa \neq 0$ (constant)

$t(s) = (\alpha(s), \beta(s))$ (say)

$\|t'(s)\|^2 = \kappa^2 = \alpha'^2 + \beta'^2$

$0 = 2\alpha'\alpha'' + 2\beta'\beta''$ — ⊛

$\Rightarrow \langle \alpha', \beta' \rangle$ & $\langle \alpha'', \beta'' \rangle$ are orthogonal

Also $\|t\| = 1 \Rightarrow \alpha^2 + \beta^2 = 1 \Rightarrow \alpha\alpha' + \beta\beta' = 0$ — ②

$\Rightarrow \langle \alpha, \beta \rangle, \langle \alpha'', \beta'' \rangle \perp \langle \alpha', \beta' \rangle$

parallel

$\mu \langle \alpha, \beta \rangle = \langle \alpha'', \beta'' \rangle$

$$\alpha'' = \mu\alpha$$
$$\beta'' = \mu\beta$$

$$② \Rightarrow \alpha'^2 + \alpha\alpha'' + \beta'^2 + \beta\beta'' = 0$$

$$K^2 + \mu\underbrace{(\alpha^2+\beta^2)}_{=1} = 0$$

$$\mu = -K^2$$

$$\left.\begin{aligned} \alpha'' + K^2\alpha = 0 \\ \beta'' + K^2\beta = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} \alpha(s) &= \sin(Ks+B) \\ \beta(s) &= \cos(Ks+B) \end{aligned}$$

<u>Def:</u>

γ_1, γ_2 (parametrized by arc-length) are said to have a contact of order K at "s_0",

if $\gamma_1(s_0) = \gamma_2(s_0)$

$\vdots$

$\gamma_1^{(K)}(s_0) = \gamma_2^{(K)}(s_0)$

Frenet curve

let $\gamma(s)$ be a regular in $\mathbb{R}^n$ (parametrized by arc-length) and n times continuously differentiable. Then γ is called a Frenet curve if for every s, $\gamma', \gamma'', \ldots, \gamma^{(n-1)}$ are linearly independent.

The Frenet-n frame $e_1, \ldots, e_n$ is then uniquely determined by the following conditions:

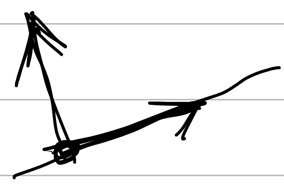
i) $e_1, \ldots, e_n$ are orthonormal
ii) For every $k = 1, \ldots, n-1$ one has

$$\text{span}\{e_1, \ldots, e_k\} = \text{span}\{\gamma', \ldots, \gamma^{(k)}\}$$

iii) $\langle \gamma^{(k)}, e_k \rangle > 0$, for $k = 1, \ldots, n-1$.

Ex. For $n=2$

$$e_1 = \gamma'$$



$$e_2 = \frac{\pi}{2} \text{ rotation c.c of } e_1$$

Now,

$$\langle \gamma', \gamma' \rangle = 1$$

$$\Rightarrow \langle \gamma'', \gamma' \rangle = 0 \Rightarrow \langle \gamma'', e_1 \rangle = 0$$

$$\Rightarrow \gamma'' = \kappa e_2$$

we have

$$\gamma' = e_1$$

$$e_1' = \gamma'' = \kappa e_2$$

$$\langle e_2, e_2 \rangle = 1$$

$$e_2' \perp e_2$$

$$e_2' = \mu e_1$$

and since

$$\langle e_1, e_2 \rangle = 0 \Rightarrow \langle e_1', e_2 \rangle + \langle e_1, e_2' \rangle = 0$$

$$\kappa + \langle e_1, \mu e_1 \rangle = 0$$

$$\mu = -\kappa$$

and we have $\begin{array}{l} e_1' = \kappa e_2 \\ e_2' = -\kappa e_1 \end{array} \Bigg| \begin{array}{l} t' = \kappa n \\ n' = -\kappa t \end{array}$

In matrix form

$$\begin{pmatrix} e_1' \\ e_2' \end{pmatrix} = \begin{pmatrix} 0 & \kappa \\ -\kappa & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

Ex! Find a parametrization of the center of osculating circle to $r(t) = \langle a\cos t, b\sin t \rangle$ at t.

★ For every regular curve in the plane with non-vanishing curvature the circle centered at

$$\gamma(s_0) + \frac{e_2(s_0)}{\kappa(s_0)}$$

with radius $\dfrac{1}{\kappa(s_0)}$ is called the

osculating circle of γ at point $\gamma(s_0)$.

Def:- A map $f: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is called a rigid motion if there exists an orthogonal matrix A with $\det(A) = 1$ and $b \in \mathbb{R}^n$ s.t

$$f(x) = Ax + b \qquad \forall x \in \mathbb{R}^n.$$

$$|A| = 1.$$

The case $n=2$ restricts to

$$f(x) = R_\varphi(x) + b$$

where $R_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$

Prop:. Let $c : [a,b] \longrightarrow \mathbb{R}^2$ be a regular curve parametrized by arc length.

Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ be a rigid motion defined by

$$f(x) = R_\varphi x + b$$

and γ be a curve defined by

$$\gamma(s) = f(c(s))$$

Then γ is a regular curve parametrized by arc-length and $K_\gamma = K_c$.

Proof:- Since $\gamma(s) = f(c(s)) = R_\rho(c(s)) + b$

$$\Rightarrow \gamma'(s) = R_\rho \cdot c'(t)$$

$Tx: Ax+b$ then, $T' = A$
$g(x) = (f \circ T)(x)$
$Dg = D_f(Tx) \cdot D_T(x)$

$c(s) = (c_1(s), c_2(s))$

$$R_\rho(c(s)) + b = \begin{pmatrix} \cos\rho & -\sin\rho \\ \sin\rho & \cos\rho \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + b$$

$$= (c_1 \cos\rho - \sin\rho\, c_2 + b_1,$$
$$c_1 \sin\rho + \cos\rho\, c_2 + b_2)$$

$$\gamma'(s) = (c_1' \cos\rho - \sin\rho\, c_2', c_1' \sin\rho + \cos\rho\, c_2')$$
$$= R_\rho(c'(s))$$

$$\langle \gamma'(s), \gamma'(s) \rangle = \langle R_\rho(c'(s)), R_\rho(c'(s)) \rangle$$
$$= R_\rho^T R_\rho \langle c'(s), c'(s) \rangle = \|c'(s)\| = 1$$

Now,

$$\kappa_\gamma(s) = \det(\gamma', \gamma'')$$

$$= \det((f\circ c)', (f\circ c)'')$$

$$= \det(R_\rho c', R_\rho c'')$$

$$= \det(R_\rho)\det(c', c'')$$

$$= \kappa_c(s)$$ ▨

Note

(Ax, Ay)

$= A(x, y)$

$\Rightarrow \det(Ax, Ay)$
$= \det(A)\det(x, y)$

Recall ::

Theorem

Given a continuous function $h: [a,b] \to \mathbb{R}$ and $s_0 \in [a,b]$, if $y: [a,b] \to \mathbb{R}$ satisfies $y' = h$ (h is the anti-derivative of y) and $y(s_0) = y_0$, then there exists a unique $y(s)$, s.t

$$y(s) = y_0 + \int_{s_0}^{s} h(t)\,dt$$

Lemma:

Given V_1, V_2 (unit vectors) in $\mathbb{R}^2$ and $p_0, q_0 \in \mathbb{R}^2$ then there exists a unique rigid motion $f = R_\rho x + b$ such that $f(p_0) = q_0$, $R_\rho V_1 = V_2$.

Proof:-

Since V_1, V_2 are unit vectors, there exists a unique angle $0 \leq \rho < 2\pi$ such that

$$V_2 = R_\rho V_1$$

To get b,

$$b = q_0 - R_\rho p_0$$

uniqueness follows by construction.

"Fundamental theorem of plane curves"

(1) Given a smooth function $K:[a,b] \to \mathbb{R}$, $P_0 = (x_0, y_0)^T \in \mathbb{R}^2$, $V_0 \in \mathbb{R}^2$ a unit vector and $s_0 \in [a,b]$, then there exists a unique $\gamma:[a,b] \to \mathbb{R}^2$ parametrized by its arc-length such that $\gamma(s_0) = P_0$, $\gamma'(s_0) = V_0$ and curvature of γ is K.

(2) If $\alpha, \beta:[a,b] \to \mathbb{R}^2$ are parametrized by arc-length and have the same curvature, then there is a unique rigid motion f of $\mathbb{R}^2$ s.t

$$\beta = f \circ \alpha$$

Proof:- Since V_0 is a unit vector, there is a constant angle θ_0 such that

$$V_0 = (\cos\theta_0, \sin\theta_0)^T$$

Set $$\theta(s) = \theta_0 + \int_{s_0}^{s} K(t)\,dt$$

$$x(s) = x_0 + \int_{s_0}^{s} \cos(\theta(t))\,dt, \qquad y(s) = y_0 + \int_{s_0}^{s} \sin(\theta(t))\,dt$$

where $\alpha(s) = (x(s), y(s))^T$

FTC $\Rightarrow$ $\theta'(s) = \kappa(s)$, $x'(s) = \cos(\theta(s))$, $y'(s) = \sin(\theta(s))$

However,

$$\alpha'(s) = (x'(s), y'(s)) = (\cos(\theta(s)), \sin(\theta(s)))$$

Since α is parametrized by its arc-length

and $\|\alpha''(s)\| = \theta'(s) = \kappa$

By lemma: $\exists!\ f(x) = R_\rho x + b$ s.t

$$f(\alpha(s_0)) = \beta(s_0)$$

$$R_\rho(\alpha'(s_0)) = \beta'(s_0)$$

let $\gamma: [a,b] \to \mathbb{R}^2$ by

$$\gamma(s) = (f \circ \alpha)(s)$$

$\Rightarrow$ γ & α have the same curvature.

Also $\gamma(s_0) = \beta(s_0)$

$$\gamma'(s_0) = R_p(\alpha'(s_0)) = \beta'(s_0)$$

also since γ & β have the same

curvature $\overset{(1)}{\Rightarrow} \gamma = \beta \Rightarrow \beta = f\circ\alpha$.

Ex! Use technology to visualize planar curves of "linear curvature" (upto rigid motion).

Space curve:

For $\underline{n=3}$, a regular three times continuously differentiable curve γ is called a Frenet curve if $\gamma'' \neq 0$ (everywhere).

The accompanying Frenet-Frame is then given by

$e_1 = \gamma'$ "tangent"

$e_2 = \dfrac{\gamma''}{\|\gamma''\|}$ "normal"

$e_3 = \gamma' \times \gamma''$ "binormal"

As before $\|\gamma''\| := \kappa$. We have

$$e_1' = \gamma'' = \kappa e_2$$

$$e_2' = \langle e_2', e_1\rangle e_1 + \underbrace{\langle e_2', e_2\rangle}_{0} e_2 + \langle e_2', e_3\rangle e_3$$

$$\langle e_2, e_1\rangle = 0 \Rightarrow \langle e_2', e_1\rangle = -\langle e_2, e_1'\rangle = -\kappa$$

$$= -\kappa e_1 + \underbrace{\langle e_2', e_3\rangle}_{:=\tau} e_3$$

$$e_2' = -\kappa e_1 + \tau e_3$$

$$e_3' = \langle e_3', e_1\rangle e_1 + \langle e_3', e_2\rangle e_2 + \underbrace{\langle e_3', e_3\rangle}_{=0} e_3$$

$$= -\underbrace{\langle e_3, e_1'\rangle}_{=0} e_1 - \underbrace{\langle e_2', e_3\rangle}_{=\tau} e_2$$

$$e_3' = -\tau e_2$$

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}' = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

osculating plane (spanned by $t(s)$ & $n(s)$)
normal plane (" " $n(s)$ & $b(s)$)
rectifying plane (spanned by $t(s)$ & $b(s)$)

$\tau = 0 \Rightarrow e_3 = \text{constant}$

$\Rightarrow$ curve is a plane curve.

" $\Longleftrightarrow$ "

<u>Remark</u>

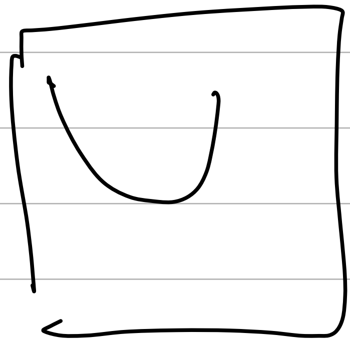
An ordinary Taylor series expansion at $s=0$ is given

$$\gamma(s) = \gamma(0) + s\gamma'(0) + \frac{s^2}{2!}\gamma''(0) + \frac{s^3}{3!}\gamma'''(0) + \dots$$

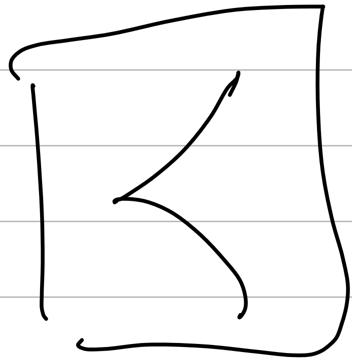
$\gamma' = e_1$, $e_1' = \gamma'' = \kappa e_2$

$\gamma''' = (\kappa e_2)' = \kappa' e_2 + \kappa e_2' = \kappa' e_2 - \kappa^2 e_1 + \kappa\tau e_3$

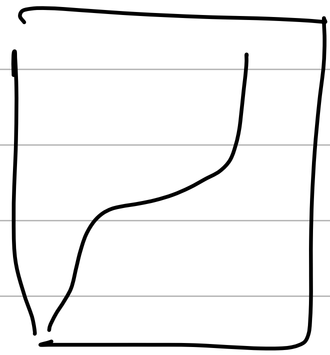
$$\gamma(s) = \gamma(0) + \left(s - \frac{s^3\kappa^2}{6}\right)e_1 + \left(\frac{s^2\kappa}{2} + \frac{s^3\kappa'}{6}\right)e_2 + \frac{s^3\kappa\tau}{6}e_3 + O(s^4)$$



osculating



normal



rectifying