Thm:- (Osculating sphere and spherical curves)

i) let γ be a Frenet curve in $\mathbb{R}^3$ with $\tau(s_0) \neq 0$

Then the surface of the sphere centered at the point

$$\gamma(s_0) + \frac{1}{\kappa(s_0)} e_2(s_0) - \frac{\kappa'(s_0)}{\tau(s_0)\kappa^2(s_0)} e_3(s_0)$$

which passes through $\gamma(s_0)$ has a contact of order 3 with $\gamma(s)$ at s_0.

This sphere is called the osculating sphere.

Proof:-

let $m(s_0)$ be the center of this sphere.

Then

$$m(s_0) - \gamma(s_0) = \alpha e_1(s_0) + \beta e_2(s_0) + \delta e_3(s_0)$$

Consider the function

$$r(s) = \langle m(s) - \gamma(s), m(s) - \gamma(s) \rangle - R_0^2$$
$$= \langle m - \gamma(s), m - \gamma(s) \rangle - R_0^2$$

Let $f: \mathbb{R}^n \to \mathbb{R}$ be smooth and assume ∇f is non-vanishing on the level set $M_c = \{f = c\}$

A smooth curve $\gamma: (s_0 - \epsilon, s_0 + \epsilon) \to \mathbb{R}^n$ has a contact of order N with M_c at $\gamma(s_0)$ if

$$f \circ \gamma : (s_0 - \epsilon, s_0 + \epsilon) \to \mathbb{R}$$

vanishes to order N.

Now $f(x) = \langle m - x, m - x \rangle - R_0^2 = \|m - x\|^2 - R_0^2 : \mathbb{R}^3 \to \mathbb{R}$

$\gamma : (s_0 - \epsilon, s_0 + \epsilon) \to \mathbb{R}^3$

sphere of radius $\|m - x\|$ has a contact of order 3 if

$$r(s) = f \circ \gamma = \langle m - \gamma, m - \gamma \rangle - R_0^2$$

vanishes upto order 3.

$r(s_0) = 0$

$r'(s_0) = \langle m - \gamma(s_0), \gamma'(s_0) \rangle = 0 \Rightarrow \alpha = 0$

$$r''(s_0)=0 \Leftrightarrow \langle m-\gamma(s_0), \gamma''(s_0)\rangle - \langle \gamma'(s_0), \gamma'(s_0)\rangle = 0$$

$$\Rightarrow \beta\kappa - 1 = 0 \Rightarrow \beta = \frac{1}{\kappa(s_0)}$$

$$r'''(s_0)=0 \Rightarrow \langle m-\gamma(s_0), \gamma'''(s_0)\rangle = 0$$

$$\Rightarrow \langle m-\gamma(s_0), \kappa' e_2 - \kappa^2 e_1 + \kappa\tau e_3\rangle = 0$$

$$\gamma' = t = e_1$$
$$\gamma'' = t' = e_1' = \kappa n$$

$$\gamma''' = e_1'' = \kappa' n + \kappa n' = \kappa' e_2 + \kappa(\tau e_3 - \kappa e_1)$$
$$= \kappa' e_2 + \kappa\tau e_3 - \kappa^2 e_1$$

$$\langle \frac{1}{\kappa} e_2 + \delta e_3, \kappa' e_2 - \kappa^2 e_1 + \kappa\tau e_3\rangle = 0$$

$$\Rightarrow \frac{\kappa'}{\kappa} + \delta\kappa\tau = 0 \Rightarrow \delta = \frac{-\kappa'}{\kappa^2\tau}$$

as required.

ii) <u>Exercise</u>:

Let γ be a smooth curve with $\tau \neq 0$ $\forall s$. Then γ lies on a sphere iff

$$\frac{\tau}{\kappa} = \left(\frac{\kappa'}{\tau \kappa^2}\right)'$$

iii) Let γ be a C^3-curve parametrized by arc-length and whose image lies on the unit sphere $S^2 \subset \mathbb{R}^3$.

Set $J := \det(\gamma, \gamma', \gamma'')$. Then γ is a Frenet curve with curvature $\kappa = \sqrt{1 + J^2}$ and torsion $\tau = \dfrac{J'}{1 + J^2}$

<u>Proof</u>:

γ, γ' independent $(\because \|\gamma\| = 1)$

$e_1 = \gamma, e_2 = \gamma', e_3 = \gamma \times \gamma''$ forms a Frenet frame

$$\Rightarrow \gamma'' = \langle \gamma'', \gamma \rangle \gamma + \langle \gamma'', \gamma' \rangle \gamma' + \langle \gamma'', \gamma \times \gamma' \rangle \gamma \times \gamma'$$

$$\langle \gamma', \gamma' \rangle = 1 \Rightarrow \langle \gamma'', \gamma \rangle = -1$$

$$\Rightarrow \gamma'' = -\gamma + J(\gamma \times \gamma')$$

$$\|\gamma''\|^2 = 1 + J^2 \Rightarrow \kappa = \sqrt{1+J^2} \quad (\Rightarrow \kappa \neq 0)$$

As for a general Frenet curve,

$$\gamma'' = \kappa e_2 \text{ or } e_2 = \frac{\gamma''}{\kappa}$$

$$e_3 = \gamma' \times e_2 = \gamma' \times \frac{\gamma''}{\kappa}$$

$$\tau = -\langle e_3', e_2 \rangle = -\langle \left(\gamma' \times \frac{\gamma''}{\kappa}\right)', \frac{\gamma''}{\kappa} \rangle$$

$$= -\left[\langle \left(\frac{\gamma'}{\kappa}\right)' \times \gamma'' + \frac{1}{\kappa}\gamma' \times \gamma''', \frac{\gamma''}{\kappa} \rangle\right.$$

$$= -\left[\langle \left(\frac{\gamma''}{\kappa} - \frac{\kappa'}{\kappa^2}\gamma'\right) \times \gamma'', \frac{\gamma''}{\kappa} \rangle + \frac{1}{\kappa^2}\langle \gamma' \times \gamma''', \gamma'' \rangle\right.$$

$$= -\frac{1}{k^2}\langle r' \times r''', r''\rangle + \frac{k'}{k^3}\underbrace{\langle r' \times r'', r''\rangle}_{=0}$$

$$= -\frac{1}{k^2}\langle r' \times r''', -r + J(r \times r')\rangle$$

$$= \frac{1}{k^2}\langle r' \times r''', r\rangle - \frac{J}{k^2}\underbrace{\langle r' \times r''', r \times r'\rangle}_{=0}$$

As $r''' \perp r$:

$\langle r, r\rangle = 1 \Rightarrow \langle r, r'\rangle = 0 \Rightarrow \langle r'', r\rangle = -1$

$\langle r''', r\rangle = 0$

$\Rightarrow\ r''' \times r' \perp r \times r'$

$\downarrow$

$\because\ (a \times b)\cdot(c \times d) = (a\cdot c)(b\cdot d) - (a\cdot d)(b\cdot c)$

$$\tau = \frac{1}{K^2} \langle \gamma' \times \gamma''', \gamma \rangle = \frac{J'}{K^2}$$

Since $J = \langle \gamma' \times \gamma'', \gamma \rangle$

$$J' = \underbrace{\langle \gamma'' \times \gamma'', \gamma \rangle}_{=0} + \langle \gamma' \times \gamma''', \gamma \rangle + \underbrace{\langle \gamma' \times \gamma'', \gamma' \rangle}_{=0}$$

$$= \langle \gamma' \times \gamma''', \gamma \rangle$$

J is called the Geodesic curvature.

$J = 0$ for great circles.

<u>Theorem</u>: For a Frenet curve in $\mathbb{R}^3$, the following are equivalent.

(i) There is a vector $v \in \mathbb{R}^3 \setminus \{0\}$ with the condition that $\langle e_1, v \rangle$ is constant.

ii) There is a vector $v \in \mathbb{R}^3 \setminus \{0\}$ with $\langle e_2, v \rangle = 0$

iii) " " " " " s.t $\langle e_3, v \rangle$ is constant

iv) The quotient $\frac{\tau}{\kappa}$ is constant.

<u>Proof.</u> For a plane curve (trivial) as $\tau=0$ and v can be chosen normal to the plane.

Assume $\tau \neq 0$

i) $\Leftrightarrow$ (ii) $\langle e_1, v\rangle = c$

$\Leftrightarrow \langle e_1', v\rangle + \langle e_1, v'\rangle = 0$

$\Leftrightarrow \kappa\langle e_2, v\rangle = 0 \Rightarrow \langle e_2, v\rangle = 0$

(iii) $\Leftrightarrow$ (ii) $\langle e_3, v\rangle = c \Leftrightarrow \langle e_3', v\rangle = 0$

$\Leftrightarrow -\tau\langle e_2, v\rangle = 0$

i), ii), iii) $\Rightarrow$ $v = \alpha e_1 + \beta e_3$

$\Rightarrow 0 = \alpha e_1' + \beta e_3'$

$0 = \alpha\kappa e_2 - \beta\tau e_2 \Rightarrow \frac{\tau}{\kappa} = \frac{\alpha}{\beta}$

Conversely, if $\frac{\tau}{\kappa} =$ constant, then

$$V := \frac{\tau}{\kappa} e_1 + e_3$$

is constant $\because$ $V' = \frac{\tau}{\kappa} e_1' + e_3'$ ☒

Example

$$\gamma(t) = \langle a\cos(\alpha t), a\sin(\alpha t), bt \rangle$$

parametrized by arc length if

$$\boxed{a^2\alpha^2 + b^2 = 1}$$

$$\gamma''(t) = \langle -a\alpha^2 \sin(\alpha t), -a\alpha^2 \cos(\alpha t), 0 \rangle$$

$$\boxed{\kappa^2 = a^2\alpha^4}$$

$$e_1 = \langle -a\alpha\sin(\alpha t), a\alpha\cos(\alpha t), b \rangle$$

$$e_2 = \langle -\sin(\alpha t), -\cos(\alpha t), 0 \rangle$$

$$e_2' = \langle -\alpha\cos(\alpha t), \alpha\sin(\alpha t), 0 \rangle$$

$e_3 = e_1 \times e_2$

$$= \begin{vmatrix} i & j & k \\ -a\alpha\sin(\alpha t) & a\alpha\cos(\alpha t) & b \\ -\sin(\alpha t) & -\cos(\alpha t) & 0 \end{vmatrix}$$

$$= i\big(b\cos(\alpha t)\big) - j\big(b\sin(\alpha t)\big) + k\big(a\alpha\cos(\alpha t)\sin(\alpha t)\big)$$

$$e_2' \cdot e_3 = -\alpha b\cos^2(\alpha t) - \alpha b\sin^2(\alpha t) = -\alpha b$$

$$\boxed{\tau^2 = \alpha^2 b^2}$$

helix $\Rightarrow$ $\frac{\kappa}{\tau}$ is constant.

Conversely, given κ, τ.

$$\kappa^2 + \tau^2 = a^2\alpha^4 + \alpha^2 b^2 = \alpha^2(a^2\alpha^2 + b^2) = \alpha^2$$

$$b^2 = \frac{\tau^2}{\kappa^2 + \tau^2}$$

$$a^2 = \frac{\kappa^2}{\alpha^4} = \frac{\kappa^2}{(\kappa^2 + \tau^2)^2}$$

$\Rightarrow$ Every Frenet curve in $\mathbb{R}^3$ with Const κ & Const τ is a part of a circular helix.

Interesting!

$$K = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}$$

$$K^2 = \begin{pmatrix} -\kappa^2 & 0 & \kappa\tau \\ 0 & -\kappa^2-\tau^2 & 0 \\ \kappa\tau & 0 & -\tau^2 \end{pmatrix}$$

$$\mathrm{Det}(K^2 - \lambda I) = -\lambda(\lambda + \kappa^2 + \tau^2)^2$$

Darboux rotation vector:

$$D = \tau e_1 + \kappa e_3$$

$e_i' = D \times e_i \qquad i = 1, 2, 3$

Darboux equations.

$e_1' = (\tau e_1 \times \kappa e_3) \times e_1 = \kappa e_2$

$\vdots$

Thm: Let γ be a Frenet curve in $\mathbb{R}^n$ with Frenet frame $e_1, \dots, e_n$. Then there are functions $\kappa_1, \dots, \kappa_{n-1}$ defined on γ with $\kappa_1, \dots, \kappa_{n-2} > 0$ so that every κ_i is $(n-1-i)$ times cont. differentiable and

$$\begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}' = \begin{pmatrix} 0 & \kappa_1 & 0 & \dots & 0 \\ -\kappa_1 & 0 & \kappa_2 & \dots & 0 \\ & \dots & \dots & \vdots & \kappa_{n-1} \\ 0 & \dots & & -\kappa_{n-1} & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}$$

κ_i is called ith Frenet curvature and the equations are called Frenet eqns.

Proof:

Consider

$$e_i' = \sum_{j=1}^{n} \langle e_i', e_j \rangle e_j$$

For $i \le n-1$, $e_i \in \text{span}\{\gamma', \dots, \gamma^{(i)}\}$

$\Rightarrow e_i = C_1(s)\gamma'(s) + \dots + C_i(s)\gamma^{(i)}(s)$

$\Rightarrow e_i'(s) = C_1'\gamma' + C_1\gamma'' + \dots + C_i'\gamma^{i} + C_i\gamma^{(i+1)}(s)$

$\Rightarrow e_i' \in \text{span}\{\gamma', \dots, \gamma^{(i+1)}\}$

$= \text{span}\{e_1, \dots, e_{i+1}\}$ — (*)

$\langle e_i', e_{i+2}\rangle = 0 = \langle e_i', e_{i+3}\rangle = \dots = \langle e_i', e_n\rangle = 0$

Define $\kappa_i = \langle e_i', e_{i+1}\rangle$

(*) $\Rightarrow \langle e_i', e_j\rangle = 0 \quad \forall j \ge i+2$

$e_2' = \langle e_2', e_1\rangle e_1 + \langle e_2', e_3\rangle e_3$

Likewise

$e_3' = \langle e_3', e_1\rangle e_1 + \langle e_3', e_2\rangle e_2 + \langle e_3', e_4\rangle e_4$
$= 0$

generally

$$e_i' = -\kappa_{i-1} e_{i-1} + \kappa_i e_{i+1}$$

$$\kappa_1 = \langle e_1', e_2 \rangle = \langle \frac{\gamma''}{\|\gamma''\|}, e_2 \rangle$$

$\because (\gamma^{(k)}, e_k) > 0.$

$$e_2 = \frac{\gamma''}{\|\gamma''\|} \Rightarrow e_2' = \frac{\|\gamma''\| \gamma''' - \gamma'' \|\gamma''\|'}{\|\gamma''\|^2}$$

$$\langle e_2', e_3 \rangle = \underset{(\alpha > 0)}{\alpha} \underbrace{\langle \gamma''', e_3 \rangle} > 0$$

$$e_3 = \frac{\gamma''' - \alpha(s) e_1 - \beta(s) e_2}{\| \quad \|}$$

ok.

$$e_3' \; \| \; \frac{\| (\gamma'''' - \alpha' e_1}{\| \quad \|^2}$$