Def:- A (C^1) curve $\gamma : [a,b] \to \mathbb{R}^2$ is called

closed if $\gamma(a) = \gamma(b)$ and $\gamma'(a) = \gamma'(b)$.

Simple if it is one to one

Convex

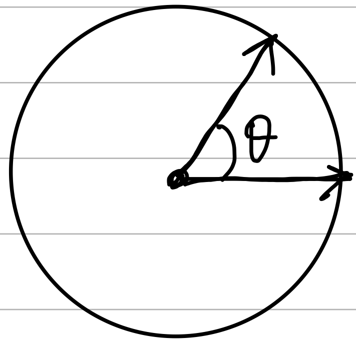
Simple closed curve that lie on one side of their tangent lines.

Recall. $t(s) = \langle \cos(\theta(s)), \sin(\theta(s)) \rangle$

$$\theta' = \kappa \quad \left[\theta(s) = \theta_0 + \int_0^s \kappa(t)\,dt \right]$$

Regard: $t : [a,b] \to S^1$

$\theta(b) - \theta(a) = 2k\pi$



Prop: If $f: [a,b] \to S^1$, cont. then $\exists$ a
$f(a) = f(b)$

continuous $\theta : [a,b] \to \mathbb{R}$ s.t

$$f(t) = \langle \cos(\theta(t)), \sin(\theta(t)) \rangle \quad \forall t \in [a,b]$$

$\theta(t)$ is unique upto adding a multiple of 2π.

i.e

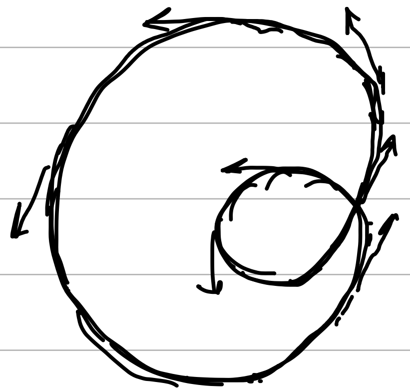
if $\varphi(t)$ is any other polar angle function then

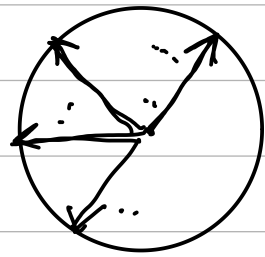
$$\varphi(t) = \theta(t) + 2k\pi$$

$$\deg(f) = \frac{\theta(b) - \theta(a)}{2\pi}$$

no. of times the domain is wrapped c.c around the circle

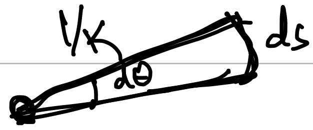
$\deg(\vec{T})$ rotation index of the curve.





Rotation index = 2.

Ex! Do it for a circle with different parametrizations.



$$ds = \frac{1}{K}\, d\theta \Rightarrow d\theta = K\, ds$$

$$\int_a^b d\theta = \int_a^b K(s)\, ds$$

$$\theta(b) - \theta(a) = \text{Total curvature}$$

$$2\pi\,(\text{rotation index}) = \text{Total curvature}$$

Prop:- Let $f_1, f_2 : [a,b] \longrightarrow S^1$ be cont. fncns with $f_1(a) = f_1(b)$ and $f_2(a) = f_2(b)$. If f_1 and f_2 have different degrees that

$$f_1(t_0) = -f_2(t_0) \text{ for some } t_0 \in [a,b].$$

Proof:- Let $\theta_1, \theta_2 : [a,b] \longrightarrow \mathbb{R}$ be the angle functions for f_1 & f_2. Consider

$$\delta(t) = \theta_2(t) - \theta_1(t)$$

Now

$$|\delta(b) - \delta(a)| = |\theta_2(b) - \theta_2(a) - (\theta_1(b) - \theta_1(a))|$$

$$\geq 2\pi$$

$\because$ degrees are different

Since $\delta(t)$ has a net-change of atleast 2π, there must be an odd integer multiple of π between $\delta(a)$ and $\delta(b)$.

Since δ is cont $\Rightarrow$ (IVT) $\exists\, t_0 \in [a,b]$ s.t

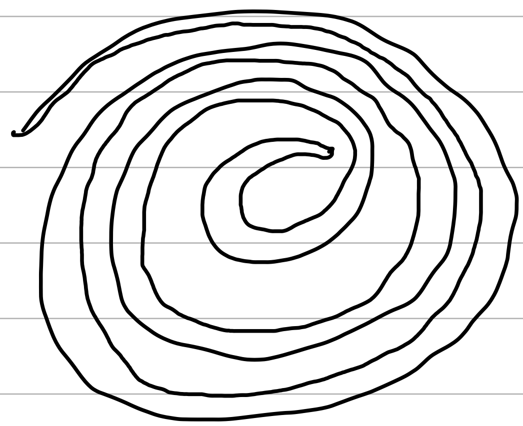
$\delta(t_0) = (2k+1)\pi, \quad k \in \mathbb{Z}$

$\Rightarrow \theta_2(t_0) = \theta_1(t_0) + (2k+1)\pi$

$\Rightarrow f_2(t_0) = -f_1(t_0).$

Hopf's Theorem

let $\gamma : [a,b] \to \mathbb{R}^2$ be a simple closed curve. The rotation index of γ is either 1 or -1.



Proof:-

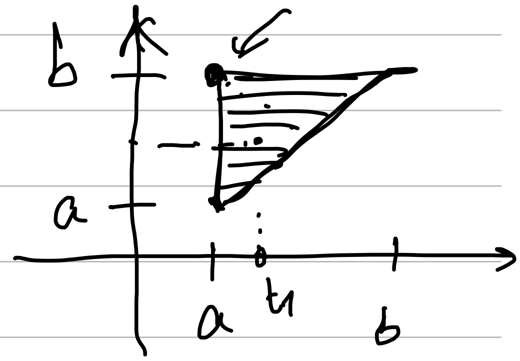
let $\gamma : [a,b] \to \mathbb{R}^2$ be a simple closed plane curve and let C be its trace.

Let $p \in C$ be a point s.t C is entirely on one side of the tangent line L.

Assume WLOG, γ is parametrized by its arc length and $\gamma(a) = p$.

Consider the triangle

$$T = \{(t_1, t_2) : a \le t_1 \le t_2 \le b\}$$



Define $\psi : T \to S^1$ as

$$\psi(t_1, t_2) = \begin{cases} \gamma'(t_1) & t_1 = t_2 \\ \dfrac{\gamma(t_2) - \gamma(t_1)}{\|\gamma(t_2) - \gamma(t_1)\|} & t_1 \ne t_2 \text{ and } (t_1, t_2) \ne (a, b) \\ -\gamma'(a) & (t_1, t_2) = (a, b). \end{cases}$$

For most inputs $\mu(t_1,t_2)$ is just a unit vector pointing in the direction from $\gamma(t_1)$ to $\gamma(t_2)$.

Rest ensure μ is continuous.

γ is a loop $\Rightarrow$

$$\lim_{t\to b-a} \frac{\gamma(t)-\gamma(a)}{b-a-t} = -\lim_{\substack{t\to b-a\\ t<(b-a)}} \frac{\gamma(b-a)-\gamma(t)}{(b-a)-t}$$

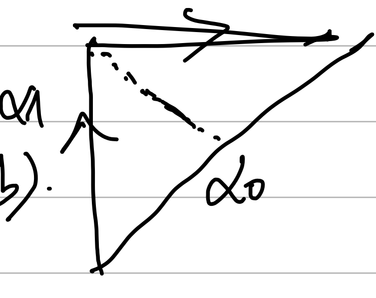
$$\boxed{\gamma(a)=\gamma(a+b-a)=\gamma(b)}$$

$$= -\gamma'(b-a)$$
$$= -\gamma'(a)$$

Let $\alpha_0:[0,1]\to T$ be a parametrization of the line segment from (a,a) to (b,b).

$\alpha_1:[0,1]\to T$

(a,a) to $(a,b) \longrightarrow (a,b) \to (b,b)$.



$\alpha_0(t) = (a - at + bt,\ a - at + bt).$

$\left(\frac{a+b}{2}, \frac{a+b}{2}\right)$

$\alpha_1(t) = \begin{cases} (a - 2at + 2at,\ a - 2at + 2bt) \\ (2a - 2at + 2bt - b,\ b) \end{cases}$

$y - \frac{a+b}{2} = \frac{\frac{a+b}{2} - b}{\frac{a+b}{2} - a} (x - \ldots$

$(1-2t)(a,a) + 2t(a,b)$

$(2-2t)(a,b) + (2t-1)(b,b)$

$y - \frac{a}{2} - \frac{b}{2} = -1\left(x - \frac{a}{2} - \frac{b}{2}\right)$

$y - \frac{a}{2} - \frac{b}{2} = -x + \frac{a}{2} + \frac{b}{2}$

$1 - 2(t - \frac{1}{2}) \quad + 2(t - \frac{1}{2})$

$y = -x + a + b$

$1 - 2t + 1$

$\forall s \in [0,1]$

$\alpha_s(t) \stackrel{?}{=}$

$(1-t)(a,a) + t(b,b)$

$\alpha_0(t) = (a - as + bs,\ a - as + bs)$

$\alpha_1(t) = \{$

$\alpha_s(t) = (1-s)\alpha_0(t) + s\,\alpha_1(t)$

$(s,t) \mapsto \alpha_s(t)$ is a cont. fncn

$[0,1] \times [0,1] \longrightarrow T$

For each $s \in [0,1]$, let $D(s)$ be the degree of

$D = \mathcal{N} \circ \alpha_s : [0,1] \longrightarrow S^1$

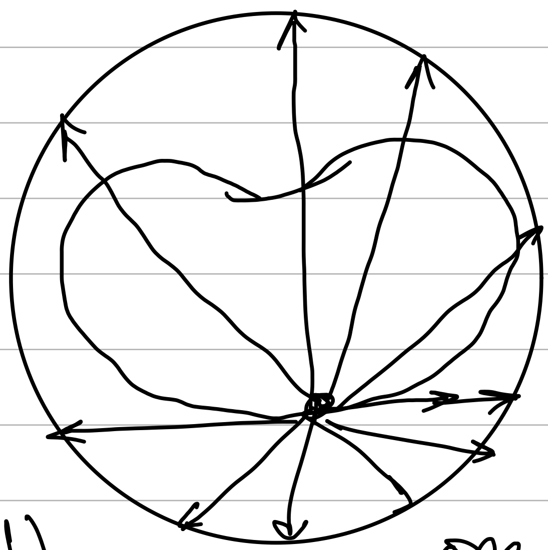
D cont + integer valued $\Rightarrow$ D must be constant

$D(0) = \mathcal{N} \circ \alpha_0$

$= \deg(\text{unit tangent of } \gamma)$

$=$ rotation index of γ

$D(0) = D(1) = \mathcal{N} \circ \alpha_1$



(a,a) to (a,b) ⟶ one Semicircle (above)

(a,b) to (b,b) ⟶ one Semi-circle (below)

$D(1) = 1$ or -1

$\boxtimes$

"Four vertex theorem"

"Every simple closed convex plane curve has atleast 4 vertices"

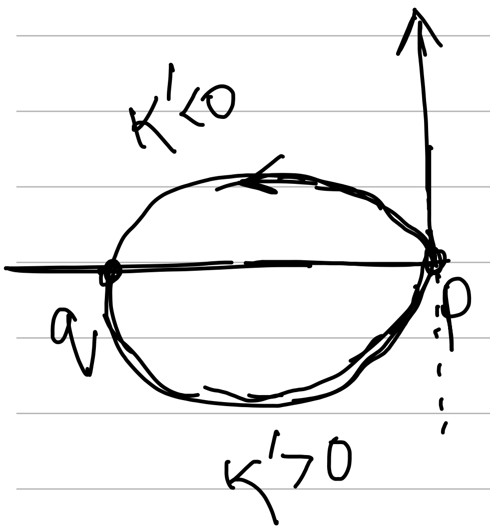
Def:- A point $\gamma(t)$ on the trace is called a vertex if signed curvature is max/min at $\gamma(t)$.

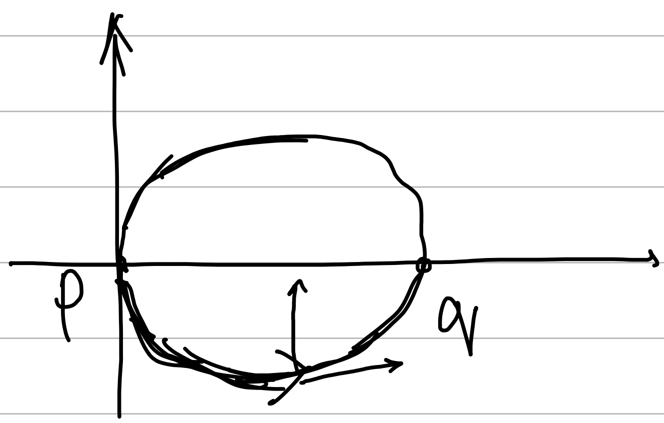
Proof. Let C be the trace.

If $\kappa_s = c$ then infinitely many vertices.

Assume vertices of γ are at $p \in C(\kappa_s \max)$ and $q \in C(\kappa_s \min)$

Let L be a line through p and q.





Choose arc-length parametrization

$\gamma : [0, L] \to \mathbb{R}^2$ s.t $\gamma(0) = p = \gamma(L)$

and $\gamma(a) = L$ for some $a \in (0, L)$.

Notice that $K_s' \leq 0$ on $(0, a)$. (max $\rightarrow$ min)

$K_s' \geq 0$ on (a, L).

WLOG, Assume p is origin and L is

x-axis.

Take $\gamma = (x, y)$, y changes sign only

at a.

In this case

$K'(t)y(t)$ does not change

sign on $[0, L]$

$$\Rightarrow \int_0^L y(t)K'(t)\,dt \neq 0.$$

but

$$y(t)K(t)\Big|_0^L - \int_0^L y'(t)K(t)\,dt$$

$$\gamma = (x, y) \Rightarrow \gamma' = (x', y')$$

$$\gamma'' = (x'', y'') = K(-y', x')$$

$$-\int_0^L x'' \, dt$$

$$= -$$