IMM-LUMS LAHORE
ADVANCED TOPOLOGY

Exam 1

Exercise 1. On $\mathbb{R}$, consider the family

$$\tau = \{U \subseteq \mathbb{R} : U \supseteq \mathbb{N}\} \cup \{\varnothing\}.$$

- Prove that τ is a topology on $\mathbb{R}$.
- For any subset $Y \subseteq \mathbb{R}$, compute its interior $\mathring{Y}$ and its closure $\overline{Y}$.
- Is $(\mathbb{R}, \tau)$ connected?
- Is $(\mathbb{R}, \tau)$ compact?
- Is the function $f\colon (\mathbb{R}, \tau) \to (\mathbb{R}, \tau)$, $f(x) = x + 1$ continuous?

Exercise 2. On $\mathbb{R}$, define

$$\delta\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \qquad \delta(x, y) = |\arctan x - \arctan y|.$$

- Prove that δ is a distance.
- Denote by d the Euclidean distance. Does there exist $c > 0$ such that $c^{-1} \cdot d(x, y) < \delta(x, y) < c \cdot d(x, y)$ for any $x, y \in \mathbb{R}$?
- Prove that δ is topologically equivalent to the Euclidean metric.

Exercise 3. Consider $X := [-1, 1] \subseteq \mathbb{R}$ with the subspace topology. Consider the equivalence relation:

$$x \sim y \quad \text{if and only if} \quad |x| = |y| < 1 \text{ or } x = y.$$

Consider the quotient $X/\sim$ with the quotient topology.

- Describe the open subsets of $X/\sim$.
- Is $X/\sim$ Hausdorff?
- Does $X/\sim$ have the T_1-separation property?

Exercise 4. Let (X, d) be a metric space. Fix $\varepsilon > 0$. We say that two points $x, y \in X$ are *ε-chain connected* if there exists a finite set of points $z_0 = x$, z_1, ..., z_{n-1}, $z_n = y$ such that $d(z_i, z_{i+1}) < \varepsilon$ for any i.

- Prove that if X is connected, then for any $\varepsilon > 0$, any points $x, y \in X$ are ε-chain connected.

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Exercise 5. For any $v \in \mathbb{R}^2$, $v \neq 0$, consider the function $\ell_v \colon \mathbb{R} \to \mathbb{R}^2$, $\ell_v(t) = tv$. Let τ be the finest topology on $\mathbb{R}^2$ that makes $\ell_v \colon (\mathbb{R}, \tau_{\text{Eucl}}) \to (\mathbb{R}^2, \tau)$ continuous for every $v \neq 0$.

- Is $(\mathbb{R}^2, \tau)$ Hausdorff?
- Does $(\mathbb{R}^2, \tau)$ admit a countable subset?
- Does $(\mathbb{R}^2, \tau)$ admit a countable basis?
- Is $(\mathbb{R}^2, \tau)$ connected or connected by path?
- Is $(\mathbb{R}^2, \tau)$ metrizable?