IMM-LUMS LAHORE 2024/2025
ADVANCED TOPOLOGY

Exam 1

Exercise 1. On $\mathbb{R}$, consider the collection of subsets that are closed under addition:

$$\mathcal{B} = \{K \subseteq \mathbb{R} : \forall x, y \in K, x + y \in K\}.$$

(a) Prove that $\mathcal{B}$ is the basis of a topology τ on $\mathbb{R}$.
(b) Describe the open sets in τ that contain only a finite number of points.
(c) Is $(\mathbb{R}, \tau)$ connected?
(d) Is $(\mathbb{R}, \tau)$ compact?
(e) Prove that the linear functions $f : (\mathbb{R}, \tau) \to (\mathbb{R}, \tau)$ given by $f(x) = \lambda \cdot x$, where $\lambda \in \mathbb{R}$, are continuous.

Solution. Note that $\mathbb{R} \in \mathcal{B}$, and that, if $K_1, K_2 \in \mathcal{B}$, then also $K_1 \cap K_2 \in \mathcal{B}$, therefore $\mathcal{B}$ is a basis for some topology on $\mathbb{R}$.
Note that, if $K \in \mathcal{B}$ contains a positive real number $\alpha > 0$, then it also contains $n \cdot \alpha$ for any $n \in \mathbb{N} \setminus \{0\}$, then it contains infinitely many points; the same argument also applies when $K \in \mathcal{B}$ contains a negative real number. Then, the only non-empty finite open set is $\{0\} \in \mathcal{B}$.
The topology τ is not connected: more precisely, $\mathbb{R} = (-\infty, 0) \cup \{0\} \cup (0, +\infty)$ with $(-\infty, 0)$, $\{0\}$, $(0, +\infty)$ disjoint clopen sets.
The topology τ is not compact: for instance, $\{\mathbb{N} \cdot x\}_{x \in \mathbb{R}}$ is a collection of open subsets that cover $\mathbb{R}$; but any finite subfamily contains at most countable many points, therefore cannot cover the whole $\mathbb{R}$.
Consider $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \lambda \cdot x$ where $\lambda \in \mathbb{R}$. To check whether it is continuous, it is enough to check whether, for every $K \in \mathcal{B}$, we have $f^{-1}(K) \in \tau$. Note that, if $x, y \in f^{-1}(K)$, then $\lambda x, \lambda y \in K \in \mathcal{B}$, therefore $\lambda x + \lambda y = \lambda(x + y) \in K$, then $x + y \in f^{-1}(K)$. Therefore, $f^{-1}(K) \in \mathcal{B}$ is open.

Exercise 2. Consider $\mathbb{R}$ endowed with the Euclidean topology, and $\mathbb{R}^{\mathbb{N}} = \{(x_n)_n : x_n \in \mathbb{R} \text{ for evey } n \in \mathbb{N}\}$, the space of real sequences, endowed with the product topology. Prove that the diagonal

$$\Delta = \{(x_n)_n : x_n = x_0 \text{ for every } n \in \mathbb{N}\} \subset \mathbb{R}^{\mathbb{N}}$$

is a closed subset. What happens if we consider the box topology on $\mathbb{R}^{\mathbb{N}}$ in place of the product topology?

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Solution. The diagonal Δ contains the constant sequences. Consider a sequence $(x_n)_n \notin \Delta$, a non-constant sequence. Therefore there exist $h \neq k$ such that $x_h \neq x_k$. Since $\mathbb{R}$ is Hausdorff, then there exist $U_h \ni x_h$, $U_k \ni x_k$ open neighbourhoods such that $U_h \cap U_k = \varnothing$. By denoting $p_i \colon \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ the ith natural projection, we have that $U := p_h^{-1}(U_h) \cap p_k^{-1}(U_k)$ is an open neighbourhood of $(x_n)_n$ (in both the product and the box topologies) such that $U \cap \Delta = \varnothing$. Therefore Δ^c is open, then Δ is closed.

Exercise 3. On $\mathbb{R}^2$, consider $\delta \colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\delta(v, w) = \begin{cases} \|v - w\| & \text{if} \quad v, w \text{ are linearly dependent,} \\ \|v\| + \|w\| & \text{if} \quad v, w \text{ are linearly independent.} \end{cases}$$

- Prove that δ is a metric.
- Denote the induced topology by τ. Is $(\mathbb{R}^2, \tau)$ connected? Is it path-connected?
- What is the topology induced on $S^1 \subseteq (\mathbb{R}^2, \tau)$ as a subspace?

Solution. It is clear that δ is non-negative and symmetric. It is also clear that $\delta(v, w) = 0$ if and only if $v = w$: indeed, if $\|v\| + \|w\| = 0$ then $v = w = 0$ but therefore they are not linearly independent; if $\|v - w\| = 0$ then $v = w$. On every line through the origin, the induced topology is the Euclidean one. In particular, the function $\alpha_v \colon [0, 1] \to \mathbb{R}^2$ defined by $\alpha(t) = t \cdot v$ for some fixed $v \in \mathbb{R}^2$ is a continuous arc with respect to τ, connecting $\alpha_v(0) = 0$ to $\alpha_v(1) = v$. In particular, τ is path-connected, then also connected.
For $\varepsilon > 0$ small enough, precisely smaller than $\|v\|$, the open neighbourhoods of v are given by open segments around v on the line though v and the origin 0. Therefore, the induced topology on S^1 is the discrete topology.

Exercise 4. On $\mathbb{R}$, consider the equivalence relation defined by: $x \sim y$ if and only if $|x| = |y|$. Consider $\mathbb{R}$ endowed with the Euclidean topology τ_{Eucl}, and the induced quotient topology on $X := (\mathbb{R}, \tau_{\text{Eucl}})/\sim$. Prove that X is homeomorphic to $[0, +\infty)$ (with the subspace topology induced by $(\mathbb{R}, \tau_{\text{Eucl}})$).

Solution. Consider the map $f \colon \mathbb{R} \to [0, +\infty)$ defined by $f(x) = |x|$. Then clearly f is continuous and surjective, and $f(x) = f(y)$ if and only if $x \sim y$. Therefore f induces a continuous, bijective function $\tilde{f} \colon X \to [0, +\infty)$. We note that $\tilde{f}$ is also open: given U open in X, namely $\pi^{-1}(U)$ is a satured open set in $\mathbb{R}$, where $\pi \colon \mathbb{R} \to X$ denotes the natural projection onto the quotient. This means that $\pi^{-1}(U)$ is a union of open intervals, symmetric with respect to the origin. Then $\tilde{f}(U) = f(\pi^{-1}(U)) = \pi^{-1}(U) \cap [0, +\infty)$ is open in $[0, +\infty)$ with respect to the subspace topology.