IMM 2024/2025
LAHORE
LUMS

ADVANCED TOPOLOGY

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TOPOLOGY ~~[illegible]~~ is a language and a theory to investigate the very shape of any geometric objects.

↳ very concrete (geometric) and very abstract (broad) too

↳ many applications:

from diff. geom and functional analysis
to topological data science and protein folding

REF. : [MUNKRES] book on topology

CONTENTS of first part:

- def, examples, constructions of top spaces
- continuous functions
- metric topology
- top properties = connectedness

ASSIGNMENT and EXAMS for the first part

- 2 assignments for the first part (see LMS)
- midterm exam
- final exam

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LECTURE 1

§12 – TOPOLOGY

DEF. A TOPOLOGY on a set X is a collection τ of subsets of X s.t.

(T1) $\phi, X \in \tau$

(T2) τ is closed under union

(T3) τ is closed under finite intersection

A subset $U \subseteq X$ is called

- OPEN if $U \in \tau$,
- CLOSED if $U^c := X \setminus U \in \tau$.

EXAMPLES
- On any set, we can define:
  - DISCRETE TOPOLOGY $\tau_{DISCR} = \mathcal{P}(X)$
  - INDISCRETE TOPOLOGY $\tau_{INDISCR} = \{\phi, X\}$
- On a two-point set $X = \{a, b\}$, we have four different topologies:

$$\begin{array}{ccc} & \tau_{INDISCR} = \{\phi, X\} & \\ \swarrow\!\!\!\cap & & \cap \\ \tau_1 = \{\phi, \{a\}, X\} & & \tau_2 = \{\phi, \{b\}, X\} \\ \cap & & \cap \\ & \tau_{DISCR} = \{\phi, \{a\}, \{b\}, X\} & \end{array}$$

Note that the function

$$f : X \to X, \quad \begin{cases} f(a) = b \\ f(b) = a \end{cases}$$

"identifies" τ_1 and τ_2

EXERCISE: How many topologies on $X = \{a, b, c\}$?

EXAMPLE (EX) On any set X, we can define the COFINITE TOPOLOGY
$$\tau_{COFIN} = \{ U : U^c \text{ is finite} \} \cup \{ \emptyset \}$$
whose ~~of~~ non-empty open sets are the complement of finite sets.

If $\# X < +\infty$, then $\tau_{COFIN} = \tau_{DISCR}$.

EXAMPLES. On $\mathbb{R}$, we have several topologies

- τ_{EUCL}: the open sets are union of open intervals (a, b)
- τ_{USC}: open sets are open intervals $(-\infty, a)$ unbounded from below
- τ_{ℓ} (LOWER-LIMIT / SORGENFREY): open sets are unions of intervals of kind $[a, b)$.

DEF. If τ, τ' are two topologies on the same set X, we say that:
τ' is FINER than τ
(or also τ is COARSER than τ')
if $\tau' \supseteq \tau$.

RMK Note that this is a PARTIAL order on topologies!

§ 13 - BASIS

DEF. If X is a set, a BASIS for a topology is a ~~subsets~~ collection of open subsets of X (called basis elements) such that: (B1) $\forall x \in X, \exists B \in \mathcal{B}$ st $x \in B$

(B2) $\forall B_1, B_2 \in \mathcal{B}$, $\forall x \in B_1 \cap B_2$, $\exists B_3 \in \mathcal{B}$ st $x \in B_3 \subseteq B_1 \cap B_2$

If $\mathcal{B}$ is a basis, we can define the TOPOLOGY GENERATED by $\mathcal{B}$ as:

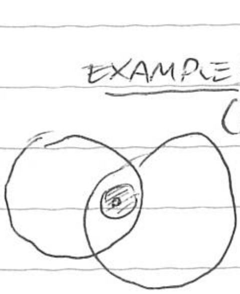
$$U \subseteq X \text{ open} \iff \forall x \in U, \exists B \in \mathcal{B} \text{ st } x \in B \subseteq U.$$

proof Thanks to (B1), we get $X \in \tau$.
Thanks to (B2), we get that τ is closed under finite intersection. □

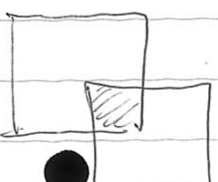
LEMMA The topology generated by a basis is the collection of unions of basis elements.

proof Any basis element is open, so also their union is. Conversely, given $U \in \tau$, $\forall x \in U$, $\exists B_x \in \mathcal{B}$ st $x \in B_x \subseteq U$. Then $U = \bigcup_{x \in U} B_x$. □

EXAMPLES (EX)



- In $\mathbb{R}^2$, we can consider the bases
$$\mathcal{B} = \{ (a,b) \times (c,d) : a<b, c<d \}$$
$$\mathcal{B}' = \{ B(x,r) : x \in \mathbb{R}^2, r>0 \}$$
We will see ~~that~~ they generate the same topology, called the Euclidean topology.



- $(\mathbb{R}, \tau_{EUCL})$ has basis $\{ (a,b) : a<b \}$, there are other bases of interest!

- $(\mathbb{R}, \tau_\ell)$ has basis $\{[a,b) : a<b\}$.
  Note that τ_ℓ is strictly finer than τ_{eucl}:
  $$(a,b) = \bigcup_{\varepsilon>0} [a+\varepsilon, b)$$

DEF. A <u>SUBBASIS</u> for a topology is a collection of open subsets whose union equals X.
The TOPOLOGY GENERATED by a subbasis S is the collection of unions of finite intersection of elements of S.
(Indeed, $B = \{$finite intersections of elements of $S\}$ is a basis.)

§ 14 – THE ORDER TOPOLOGY

Let X be a set with a SIMPLE ORDER relation (ie: the order is total: $\forall x, y$ either $x<y$ or $y<x$; nonreflexive: $\nexists\, x$ st $x<x$; transitive: $\forall x,y,z$, $x<y$ and $y<z \Rightarrow x<z$.)
Define the intervals

$$(a,b) = \{x : a<x<b\},$$

and $[a,b)$, $(a,b]$, $[a,b]$ similarly.

DEF. Let X be a set (with at least two points), endowed with a simple order.
Define $\mathcal{B}$ the collections of subsets of type:

(1) all open intervals (a,b);

(2) if $\exists\, a_0$ = smallest element of X, also all intervals of type $[a_0, b)$;

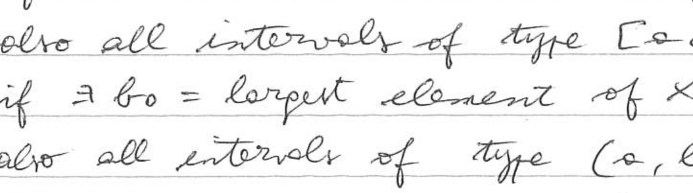
(3) if $\exists\, b_0$ = largest element of X, also all intervals of type $(a, b_0]$.

One checks that $\mathcal{B}$ is a basis, whose generated topology will be called ORDER TOPOLOGY.

EX. • The order top on $(\mathbb{R}, <)$ is the Euclidean top.

• The order top on $(\mathbb{N}, <)$ is the discrete top $\left(\begin{matrix} (n-1, n+1) = \{n\} \\ [0,1) = \{0\} \end{matrix}\right)$.

• In $\mathbb{R}^2$ with the lexicographic order ($a\times b < c\times d$ if $a<c$ or $a=c$ and $b<d$), the basis elements ~~of type~~ for the order top are of type



Indeed, the collection of subsets of the second type only is also a basis.

• In $\{1,2\}\times\mathbb{N}$ with the lexicographic order, the order top is NOT discrete (Indeed, $\{2\times 0\}$ is not ~~closed~~ open; every other point is open!).

RMK • The open rays $(-\infty, a)$, $(a, +\infty)$ form a subbasis for order top.

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§15 - PRODUCT TOPOLOGY

Let X, Y be top space and consider the Cartesian product
$$X \times Y = \{ (x,y) : x \in X, y \in Y \}.$$

DEF. The PRODUCT TOPOLOGY on $X \times Y$ is the top generated by the basis
$$\mathcal{B} = \{ U \times V : U \in \tau_X, V \in \tau_Y \}.$$

(Note that $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$.)

THM If $\mathcal{B}$ is a basis for X, $\mathcal{C}$ is a basis for Y, then $\{ B \times C : B \in \mathcal{B}, C \in \mathcal{C} \}$ is a basis for $X \times Y$.

proof We note that:

LEMMA Let (X, τ) be a top. space. If $\mathcal{C}$ is a collection of subsets of X st:
$$\forall U \in \tau, \forall x \in U, \exists C \in \mathcal{C} \text{ st } x \in C \subseteq U,$$
then $\mathcal{C}$ is a basis and generates τ.

Take W open in $X \times Y$. Fix $(x,y) \in W$. Then $\exists U \times V \in \mathcal{B}_{X \times Y}$ st $(x,y) \in U \times V \subseteq W$, where $U \subset X$, $V \subset Y$ open. Since $\mathcal{B}$ is a basis for X, then $\exists B \in \mathcal{B}$ st $x \in B \subseteq U$. Since $\mathcal{C}$ is a basis for Y, then $\exists C \in \mathcal{C}$ st $y \in C \subseteq V$. Then $(x,y) \in B \times C \subset U \times V \subseteq W$.

$\square$

EXAMPLES The product $(\mathbb{R}, \tau_{eucl}) \times (\mathbb{R}, \tau_{eucl})$ is generated by $\mathcal{B} = \{ (a,b) \times (c,d) \}$: EUCLIDEAN TOP.

DEF. The CANONICAL PROJECTIONS are:

$$\pi_1 : X \times Y \to X, \quad \pi_1(x,y) = x$$
$$\pi_2 : X \times Y \to Y, \quad \pi_2(x,y) = y.$$

THM $\mathcal{S} := \{\pi_1^{-1}(U) : U \in \tau_X\} \cup \{\pi_2^{-1}(V) : V \in \tau_Y\}$
is a subbasis for the product topology.

proof. • Indeed, $\pi_1^{-1}(U) = U \times Y \in \tau$, $\pi_1^{-1}(V) = X \times V \in \tau$
so the top generated by $\mathcal{S}$ is COARSER than the product topology.

• Moreover, the basis elements for the product topology can be written as:

$$U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V).$$

§ 16 – SUBSPACE TOPOLOGY

Let X be a top space and $Y \subseteq X$ a subset.

DEF. The SUBSPACE TOPOLOGY on Y is:

$$\tau_Y := \{ U \cap Y : U \in \tau \}.$$

RMKS: • If $\mathcal{B}$ is a basis for X, then

$\mathcal{B}_Y = \{ B \cap Y : B \in \mathcal{B} \}$ is a basis for Y.

• If $A \subseteq X$, $B \subseteq Y$ subspaces, then $A \times B \subseteq X \times Y$ subspace.

EX. $Y = [0,1] \subseteq (\mathbb{R}, \tau_{\text{eucl}})$. A basis for Y is:

$$(a,b) \cap Y = \begin{cases} (a,b) & \text{if } 0 \leq a < b \leq 1 \\ [0,b) & \text{if } a < 0 < b \leq 1 \\ (a,1] & \text{if } 0 \leq a \leq 1 < b \\ [0,1] & \text{if } a < 0 < 1 < b \\ \emptyset & \text{otherwise} \end{cases}$$

§ 17 - CLOSED SETS AND LIMIT POINTS

Recall that a subset $A \subset X$ is called CLOSED if $A^c \in \tau$.
Note that subsets can be either open, or closed, or both, or neither!

EX.
- $[a,b]$ and $[a,+\infty)$ are closed in $(\mathbb{R}, \tau_{EUCL})$.
- Every subsets is both open and closed in (X, τ_{DISCR})
- In (X, τ_{COFIN}), finite subsets are closed.
- ~~[a,b)~~ $(a,b]$ is neither open or closed in $(\mathbb{R}, \tau_{EUCL})$

DEF. Given a subset $A \subseteq X$, we define:

- INTERIOR of A:
$$int(A) = \mathring{A} = \bigcup_{\substack{U \subseteq A \\ U \text{ open}}} U$$
the largest open set contained in A

- CLOSURE of A:
$$clos(A) = \overline{A} = \bigcap_{\substack{C \supseteq A \\ C \text{ closed}}} C$$
the smallest closed set containing C.

Note that $\mathring{A} \subseteq A \subseteq \overline{A}$, and
$$A \text{ open} \iff A = \mathring{A}, \qquad A \text{ closed} \iff A = \overline{A}.$$

THM Let $A \subseteq X$ be a subset. Then:
$$x \in \overline{A} \iff \forall U \text{ open set}, U \ni x, \quad U \cap A \neq \emptyset.$$

proof
- If $x \notin \overline{A}$, then $X \setminus \overline{A}$ is open set containing x and disjoint from A.
- If $\exists U$ open, $U \ni x$, $U \cap A = \emptyset$, then $X \setminus U$ is closed containg A, then $X \setminus U \supseteq \overline{A}$, then $x \notin \overline{A}$. $\square$

DEF. We say that $U \subseteq X$ is an (OPEN) NEIGHBOURHOOD of x in X if: U is open and $U \ni x$.

DEF. Let $A \subset X$ be a subset of a top space.
We say that x is a LIMIT POINT (or POINT OF ACCUMULATION) for A if:
$$\forall\, U \ni x \text{ neigh}, \quad (U \cap A) \setminus \{x\} \neq \emptyset.$$
Set $A' := \{\text{limit points of } A\}$.

EX
- $A = (0,1] \subset \mathbb{R} \rightsquigarrow \overline{A} = [0,1]$, $A' = [0,1]$
- $B = \{ 1/n : n \in \mathbb{N} \setminus \{0\}\} \subset \mathbb{R} \rightsquigarrow \overline{B} = B \cup \{0\}$, $B' = \{0\}$
- $\mathbb{Q} \subset \mathbb{R} \rightsquigarrow \overline{\mathbb{Q}} = \mathbb{R}$, $\mathbb{Q}' = \mathbb{R}$.

THM $\overline{A} = A \cup A'$.
In particular, A is closed iff it contains all its limit points.

Proof: ($\supseteq$) is clear.
($\subseteq$) if $x \in \overline{A}$ and $x \notin A$, then $\exists\, U \ni x$ neigh that intersects $A = A \setminus \{x\}$. $\square$

DEF. X is called <u>HAUSDORFF</u> (or T_2) if: $\forall x \neq y$, $\exists\, U \ni x$ neigh, $\exists\, V \ni y$ neigh st $U \cap V = \emptyset$.

THM If X is Hausdorff, then it satisfies: <u>T_1-PROP.</u>: every point is closed.

proof $\{x\}$ closed $\iff$ $X \setminus \{x\}$ open $\square$

RMK The converse does not hold! cofinite top on an infinite set.

DEF. A sequence $\{x_n\}_n \subset X$ <u>**CONVERGES**</u> to a limit $x \in X$ if: $\forall\, U \ni x$ neigh, $\exists\, N \in \mathbb{N}$ st $\forall n \geq N$, $x_n \in U$.

THM If X is Hausdorff, then any sequence converges to <u>at most</u> one limit.

EX. $X = \{a, b, c\}$
$\tau = \{\emptyset, X, \{a, b\}, \{b, c\}, \{b\}\}$

The sequence $\{x_n = b\}_n$ converges to a, b, c!

TUTORIAL 1

EX Let X be a set. Prove that the collection
$$\tau := \{ A \subseteq X : X \setminus A \text{ is finite} \} \cup \{\emptyset\}$$
is a topology (called cofinite topology).
What is τ when X is finite?

LEMMA Let (X,τ) be a top. space. Consider the collection of closed subsets: $\sigma = \{ F \subseteq X : X \setminus F \in \tau \}$.
Prove that

(C1) $\emptyset, X \in \sigma$

(C2) σ closed under intersection

(C3) σ closed under finite union

Conversely, given a set X and a collection σ satisfying (C1), (C2), (C3), then there exists a unique topology τ on X st σ is the collection of closed subsets of (X,τ) (it is defined as $\tau := \{ A \subseteq X : X \setminus A \in \sigma \}$.)

EX In $\mathbb{R}^2$, prove that
$$\mathcal{B} = \{ B(x,r) : x \in \mathbb{R}^2, \ r > 0 \}$$
is a basis.

LEMMA For any $y \in B(x,r)$, there exits $r' > 0$ such that $y \in B(y,r') \subseteq B(x,r)$.

EX In $\mathbb{R}^2$, consider the topologies

- τ generated by $\mathcal{B} = \{ B(x,r) : x \in \mathbb{R}^2, \ r > 0 \}$
- τ' generated by $\mathcal{B}' = \{ (a,b) \times (c,d) : a < b, \ c < d \}$

Prove that $\tau = \tau'$.

LEMMA. Prove that τ' finer than $\tau \iff \forall x \in X, \forall B \in \mathcal{B}, B \ni x, \exists B' \in \mathcal{B}'$ st $x \in B' \subseteq B$.

Ex • In $\mathbb{R}$, prove that
$$\mathcal{B} = \{ (a,b) : a<b, \quad a,b \in \mathbb{Q} \}$$
is a basis that generates τ_{eucl}. (It is COUNTABLE!!)

• In $\mathbb{R}$ with τ_e, is it true that $\mathcal{B} = \{(a,b) : a,b \in \mathbb{Q}\}$ is a basis?

Ex Let X, Y be top spaces. Let $A \subseteq X$, $B \subseteq Y$ endowed with the subspace top, and consider $A \times B$ with the product top. Show that $A \times B \subseteq X \times Y$ is a subspace.
(Note that the general basis element for the subspace top $(U \times V) \cap (A \times B)$ is equal to the general basis element for the product top $(U \cap A) \times (V \cap B)$.)

Ex Let $Y \subseteq X$ be a subspace. For any $A \subseteq Y$, we have
$$\overline{A}^Y = \overline{A}^X \cap Y.$$
(Note that $\overline{A}^X$ is closed in X, then $\overline{A}^X \cap Y$ is closed in Y, containing A, then $\overline{A}^X \cap Y \supseteq \overline{A}^Y$.
Conversely, $\overline{A}^Y$ is closed in Y, so $\overline{A}^Y = C \cap Y$ for some closed set C in X, with $C \supseteq \overline{A}^X$, then $\overline{A}^X \cap Y \subseteq \overline{A}^Y$. $\square$)

Ex • Show that products of Hausdorff spaces is still Hausdorff.
• Show that a subspace of Hausdorff spaces is still Hausdorff.

Ex Consider $\mathbb{R}$ and an element $\infty \notin \mathbb{R}$, set $X := \mathbb{R} \cup \{\infty\}$.
Prove that the collection of subsets of X containing

(i) $U \subseteq \mathbb{R}$ that are open

(ii) $U \ni \infty$ st $\mathbb{R} \setminus U$ is closed and bounded

is a topology on X.

Ex On $\mathbb{Z}$, define the topology where the neighbourhood of a point $a \in \mathbb{Z}$ are:

$$N_{a,b} := \{ a + nb : n \in \mathbb{Z} \} \qquad \text{for } b \in \mathbb{Z},\ b > 0.$$

- Show that τ is a top. (GOLOMB TOPOLOGY)
- Is $\{-1, 1\}$ open? ~~[illegible]~~
- Prove that prime numbers are infinite.

($N_{a,b} \cap N_{a,b'} = N_{a, \mathrm{lcm}\{b,b'\}}$.

Every $N_{a,b}$ is open and closed, since $\mathbb{Z} \setminus N_{a,b} = \bigcup_{i=0}^{a-1} N_{i,b}$.

Every non-empty set is infinite, then $\{-1,1\}$ is not open.

If prime numbers are finite, then $\mathbb{Z} \setminus \{-1,1\} = \bigcup_{p \text{ prime}} N_{0,p}$ would be closed.)

Ex In $\mathbb{R}^2$, prove that $\tau = \big\{ \{(x,y) : xy = a\} : a \in \mathbb{R} \big\}$ is a top.

Compute $\mathring{A}$ for $A = \{ x^2 + y^2 = 1 \}$.

Ex For $A, B \subseteq X$, prove

$$\overline{A \cup B} = \overline{A} \cup \overline{B}, \qquad \overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$$

Give an example for which $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$.

(Note that, for $A \subseteq B$, we have $\mathring{A} \subseteq A \subseteq B \subseteq \overline{B}$. In particular, $\mathring{A}$ is open contained in B, then $\mathring{A} \subseteq \mathring{B}$. Similarly $\overline{A} \subseteq \overline{B}$.

Now, $A \subseteq A \cup B$, $B \subseteq A \cup B$, then $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$. Conversely, $\overline{A} \cup \overline{B}$ is a closed set containing $A \cup B$, so $\overline{A} \cup \overline{B} \supseteq \overline{A \cup B}$.

Consider, in $(\mathbb{R}, \tau_{eucl})$: $A = (-\infty, 0)$, $B = (0, +\infty)$, with $A \cap B = \emptyset$ but $\overline{A} = (-\infty, 0]$, $\overline{B} = [0, +\infty)$ with $\overline{A} \cap \overline{B} = \{0\}$.)

or also $\overline{\mathbb{Q}} \cap \overline{(\mathbb{R} \setminus \mathbb{Q})} = \mathbb{R}$, while $\mathbb{Q} \cap (\mathbb{R} \setminus \mathbb{Q}) = \emptyset$.

EX Let X be a set, $x_0 \in X$ fixed. Prove
ASS
$$\tau_1 = \{A \subseteq X,\ x_0 \in A\} \cup \{\emptyset\},$$
$$\tau_2 = \{A \subseteq X,\ x_0 \notin A\} \cup \{X\},$$
are topologie, and (X, τ_1) is not homeom to (X, τ_2).

(In τ_2, tutti i punti $\{x\}$, $x \neq 0$, sono aperti, mentre l'unico punto aperto in τ_1 è $\{0\}$.)

EX For $A \subseteq X$, $B \subseteq Y$ subspaces, prove $\overline{A \times B} = \overline{A} \times \overline{B}$.
ASS

(Note $(x,y) \in \overline{A \times B}$ iff every neighb of (x,y) in ~~X×Y~~ intersects $A \times B$. Such neighb contains some neighb of the form $U \times V$, with $U \ni x$ neigh in X, $V \ni y$ neigh in Y, so ~~x~~ $x \in \overline{A}$, $y \in \overline{B}$.)

EX Prove X Hausdorff iff $D = \{(x,x) : x \in X\} \subseteq X \times X$ closed.
ASS

(~~Let~~ $(x,y) \notin D$, ie $x \neq y$. ~~#~~ If X Hausdorff, then $\exists U \ni x$ neigh, $\exists V \ni y$ neigh, ~~st~~ $U \cap V = \emptyset$. Then $U \times V \ni (x,y)$ neigh st $(U \times V) \cap D = \emptyset$. So D closed. Converse similar.)

LECTURE 2

§ 18 - CONTINUOUS FUNCTIONS

DEF. Let X, Y be top spaces, $f: X \to Y$ function is called CONTINUOUS if: $\forall\, V \in \tau_Y$, $f^{-1}(V) \in \tau_X$.

RMK Since $f^{-1}(\bigcup_\alpha V_\alpha) = \bigcup_\alpha f^{-1}(V_\alpha)$ and $f^{-1}(\bigcap_\alpha V_\alpha) = \bigcap_\alpha f^{-1}(V_\alpha)$, it is enough to check continuity on the elements of a basis/subbasis.

EXAMPLE. A function $f: (\mathbb{R}, \tau_{eucl}) \to (\mathbb{R}, \tau_{eucl})$ is continuous (in the sense of the previous definition) if and only if it is continuous in the analytic sense:

$\forall x_0 \in \mathbb{R}, \forall \varepsilon > 0, \exists \delta > 0$ st $\forall x$ st $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$.

proof. ($\Rightarrow$) given $x_0 \in \mathbb{R}$, $\varepsilon > 0$, consider $V := (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$. It is open, then $f^{-1}(V)$ is open containing x_0, then $\exists (a, b)$ st $x_0 \in (a, b) \subseteq f^{-1}(V)$. Take $\delta := \min\{x_0 - a, b - x_0\}$, then $x_0 \in (x_0 - \delta, x_0 + \delta) \subseteq (a, b) \subseteq f^{-1}(V)$.

($\Leftarrow$) Let $V \subset \mathbb{R}$ open. If $f^{-1}(V) = \emptyset$, then it is open. Otherwise, take $x_0 \in f^{-1}(V)$. Then $f(x_0) \in V$, then $\exists\, \varepsilon > 0$ st $(f(x_0) - \varepsilon, f(x_0) + \varepsilon) \subseteq V$. Then $\exists\, \delta > 0$ st $\forall x \in (x_0 - \delta, x_0 + \delta)$ $f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon) \subseteq V$, that is, $(x_0 - \delta, x_0 + \delta) \subseteq f^{-1}(V)$.

$\square$

RMK Continuity depends on the topology!

$id: \mathbb{R}_{eucl} \to \mathbb{R}_\ell$ is NOT continuous,

but $id: \mathbb{R}_\ell \to \mathbb{R}_{eucl}$ is continuous!

THM For $f: X \to Y$ between top spaces, the following are equivalent:

(•) f continuous (namely, $\forall V \in \tau_Y$, $f^{-1}(V) \in \tau_X$)

(•) $\forall A$, $f(\overline{A}) \subseteq \overline{f(A)}$

(•) $\forall F \subseteq Y$ closed, $f^{-1}(F) \subseteq X$ closed

(•) $\forall x \in X$, f is continuous at x, ie

$\forall V \ni f(x)$ neighb, $\exists U \ni x$ neigh st $f(U) \subseteq V$.

THM (•) constant functions are cont

(•) for $Y \subseteq X$ subsp, $i: Y \hookrightarrow X$ is cont

(•) if $f: X \to Y$, $g: Y \to Z$ cont, then $g \circ f: X \to Z$ cont

(•) if $f: X \to Z$ cont, $Y \subseteq Z$ subspace, then $f|_Y: Y \to Z$ cont

(•) if $f: X \to Y$ cont, Z st $f(X) \subseteq Z \subseteq Y$ subspace, then $f: X \to Z$ cont

(•) if $f: X \to Y$ where $X = \bigcup_\alpha U_\alpha$, U_α open, $f|_{U_\alpha}: U_\alpha \to Y$ cont, then $f: X \to Y$ cont

proof (•) $f^{-1}(V) = \emptyset$ or $f^{-1}(V) = X$

(•) $i^{-1}(U) = U \cap Y$

(•) $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$

(•) $f|_Y = f \circ i$

(•) V open in $Z \iff V = U \cap Z$ with U open in Y. Then $f^{-1}(V) = f^{-1}(U \cap Z) = f^{-1}(U)$.

(•) $f^{-1}(V) = \bigcup_\alpha [f^{-1}(V) \cap U_\alpha]$

$= \bigcup_\alpha (f|_{U_\alpha}^{-1}(V))$.

↳ open in U_α, but U_α open in X, then also open in X □

THM (PASTING LEMMA). Let $X = A \cup B$ with A, B closed.

If $f: A \to Y$, $g: B \to Y$ are cont functions st $f(x) = g(x)$ for any $x \in A \cap B$, then

$h: X \to Y$, $h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}$ is cont.

proof Let C be closed in Y, then $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$. □

EX Let $\alpha: [0,1] \to X$ cont (it's called ARC), $\beta: [0,1] \to X$ cont, st $\alpha(1) = \beta(0)$. Then the joint

$$(\alpha * \beta)(t) := \begin{cases} \alpha(2t) & \text{if } 0 \le t \le 1/2 \\ \beta(2t-1) & \text{if } 1/2 \le t \le 1 \end{cases}$$

is still cont (an arc).

DEF. A function $f: X \to Y$ st:

f bijective, f cont, f^{-1} cont

is called <u>homeomorphism</u>.

Namely, U open in X iff $f(U)$ open in Y: there is a bijective correspondence between open sets.

DEF. A <u>TOPOLOGICAL PROPERTY</u> is a property entirely expressed in terms of the topology of X, namely, invariant by homeomorphism.

EX.
- $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 3x+1$ is a homeomorphism, with inverse $f^{-1}(y) = \frac{1}{3}y - \frac{1}{3}$
- $id: \mathbb{R}_\ell \to \mathbb{R}$ is cont and bij, but <u>not</u> homeom
- $f: [0,1) \to S^1$, $f(t) = e^{2\pi\sqrt{-1}t} = (\cos(2\pi t), \sin(2\pi t))$ is <u>not</u> homeom.

DEF. A function $f: X \to Y$ st

f inj, f cont, $f: X \to f(X)$ homeom

is called <u>embedding</u>.

§ 19 – PRODUCT TOPOLOGY

Given $\{X_\alpha\}_{\alpha\in J}$ a family of sets, we consider $X := \bigcup X_\alpha$ and define the cartesian product

$$\prod_\alpha X_\alpha := \{ x = (x_\alpha)_\alpha : J \to X \quad \text{st} \quad x_\alpha = x(\alpha) \in X_\alpha \}.$$

We also define the canonical projections

$$\pi_\beta : \prod_\alpha X_\alpha \to X_\beta, \quad \pi_\beta(x) = x(\beta).$$

DEF. On $\prod_\alpha X_\alpha$, we have ~~two~~ topologies:

- The box topology is generated by the basis
$$\mathcal{B} = \{ \prod_\alpha U_\alpha \; : \; U_\alpha \subseteq X_\alpha \text{ open} \}$$

- The product topology is generated by the subbasis
$$\mathcal{S} = \bigcup_{\beta\in J} \{ \pi_\beta^{-1}(U_\beta) \; : \; U_\beta \subseteq X_\beta \text{ open} \}$$

RMK A basis for the product top is

$$\left\{ B = \pi_{\beta_1}^{-1}(U_{\beta_1}) \cap \dots \cap \pi_{\beta_n}^{-1}(U_{\beta_n}) \; : \; \begin{array}{l} n\in\mathbb{N},\ \beta_1,\dots,\beta_n \in J, \\ U_{\beta_j} \subseteq X_{\beta_j} \text{ open} \end{array} \right\}$$

namely elements of the form

$B = \prod U_\alpha$ where $U_\alpha \subseteq X_\alpha$ open and $U_\alpha = X_\alpha$ except for finitely-many α.

Therefore: product topology is coarser than box top!

RMKS (•) if X_α Hausdorff, then $\prod X_\alpha$ Hausdorff, both for box top and product top

(•) take subsets $A_\alpha \subseteq X_\alpha$. Then $\overline{\prod A_\alpha} = \prod \overline{A_\alpha}$, both for box top and product top.

The product top is characterized by the following universal property.

THM A function $f: Y \to \prod_\alpha X_\alpha$, $f(x) = (f_\alpha(x))_\alpha$, where we denote $f_\alpha : Y \to X_\alpha$ the components, satisfies:

$$f: Y \to \prod X_\alpha \text{ is cont} \iff \forall \alpha,\ f_\alpha : \cancel{X} \to X_\alpha \text{ is cont.}$$

TUTORIAL 2

Ex Prove that $(a,b) \simeq (0,1) \simeq \mathbb{R}$. But $[0,1] \not\simeq \mathbb{R}$.

(Take $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\frac{x-a}{b-a}$. Then $f|_{(a,b)}:(a,b)\to(0,1)$ homeom.
Consider $g:(0,+\infty)\to\mathbb{R}$, $g(x)=\lg(x)$ homeom, inverse $g^{-1}(y)=e^{y}$.
Consider $h:(0,+\infty)\to(0,1)$, $h(x)=\frac{2}{\pi}\operatorname{atan} x$, inverse $h^{-1}(y)=\tan(\frac{\pi}{2}y)$.
If $\exists\ \psi:[0,1]\to\mathbb{R}$, then $\psi|_{[0,1]-\{0\}}:(0,1]\to\mathbb{R}\setminus\{\psi(0)\}$ still homeom, but $\mathbb{R}\setminus\{\psi(0)\}=\underbrace{(-\infty,\psi(0))}_{A:=}\cup\underbrace{(\psi(0),+\infty)}_{B:=}$,
where A,B are open disjoint sets,
therefore $\psi^{-1}(A),\psi^{-1}(B)$ are open disjoints sets covering $[0,1]$,
impossible!)

Ex Let $f:X\to Y$ cont. Prove $f(\overline{A})\subseteq\overline{f(A)}$.
(Take $x\in\overline{A}$, $\forall U\ni f(x)$ neigh. Then $f^{-1}(U)\ni x$ neigh. Then $f^{-1}(U)\cap A\neq\emptyset$, then $f(f^{-1}(U)\cap A)=U\cap f(A)\neq\emptyset$.)

Ex The stereographic projection

$$f: S^n\setminus\{N\}\to\mathbb{R}^n$$

is a homeomorphism defined by

$$f(x_0,\dots,x_n)=\frac{1}{1-x_0}(x_1,\dots,x_n)$$

= line connecting $N=(1,0,\dots,0)$ to $x=(x_0,x_1,\dots,x_n)$ intersected to hyperplane $\{x_0=0\}\simeq\mathbb{R}^n$

with inverse

$$f^{-1}(y_1,\dots,y_n)=\left(1-\frac{2}{1+y_1^2+\dots+y_n^2},\ \frac{2y_1}{1+y_1^2+\dots+y_n^2},\ \dots,\ \frac{2y_n}{1+y_1^2+\dots+y_n^2}\right).$$

Ex A subset $D\subseteq X$ is called <u>DENSE</u> if $\overline{D}=X$.
ASS
Prove that D is dense iff $D\cap A\neq\emptyset$ for every open sets $A\subseteq X$.
Prove that, if $D\subseteq X$ dense, if $f:X\to Y$ cont, surj, then $f(D)\subset Y$ dense.

(Recall the projection formula $f(A\cap f^{-1}(B))=f(A)\cap B$.)

EX Let $f, g : X \to Y$ cont, with values in Y Hausdorff.
Prove that $Z := \{ x \in X : f(x) = g(x) \}$ is closed.
In particular, if $D \subseteq X$ is dense such that $f|_D = g|_D$, then $f = g$.

EX. Let $f : \mathbb{R} \to \mathbb{R}$ cont. s.t.:
$$\forall x, y, \qquad f(x+y) = f(x) + f(y).$$
Prove that $f(x) = Kx$ for some $K = f(1) \in \mathbb{R}$.
(By $f(0) = f(0) + f(0) = 2 f(0)$ we get $f(0) = 0$.
Set $K := f(1)$. By induction, $f(n) = Kn$ for $n \in \mathbb{N}$.
Then $0 = f(0) = f(z + (-z)) = f(z) + f(-z)$, then $f(z) = Kz$
for $z \in \mathbb{Z}$. By $f(n) = f(m \frac{n}{m}) = m f(\frac{n}{m})$ we get the statement
for $\frac{n}{m} \in \mathbb{Q}$. Lastly, note that $\mathbb{Q}$ is dense in $\mathbb{R}$.)

EX Compute the cardinality of $C(\mathbb{R}; \mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} \text{ cont} \}$.

(Constant functions are continuous, so $\# C(\mathbb{R}; \mathbb{R}) \geq \# \mathbb{R}$.
$\mathbb{Q}$ is countable and dense, and $\mathbb{R}$ is Hausdorff, so
the map $C(\mathbb{R}, \mathbb{R}) \to \mathbb{R}^{\mathbb{Q}}$ is injective. Therefore $\# C(\mathbb{R}; \mathbb{R}) \leq \# \mathbb{R}^{\mathbb{Q}} = \# \mathbb{R}$.
We conclude by CANTOR-SCHRÖDER-BERNSTEIN.)

RECALL: $\# A \leq \# B \overset{\text{def}}{\iff} \exists\, A \hookrightarrow B$ injective map.

EX Construct a function which is continuous only at one point.

(Take $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$.)

EX Let $f : X \to Y$, define its graph as
$$\Gamma_f := \{ (x, y) : x \in X, y = f(x) \} \subseteq X \times Y.$$
Prove f cont $\iff$ $k : X \to \Gamma_f$ homeom.
$$k(x) = (x, f(x))$$

Ex Let $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} x & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{Q} \end{cases}$.
Let τ be the coarsest topology such that $f: (X, \tau) \to (X, \tau_{Eucl})$ is continuous. What is $\overline{\{1\}}$? What is $\overline{\{\sqrt{2}\}}$?

($\overline{\{q\}} = \mathbb{Q}$ for any $q \in \mathbb{Q}$, $\overline{\{x\}} = \{x\}$ for any $x \in \mathbb{R} \setminus \mathbb{Q}$.)

Ex. Let X, Y be top spaces, fix $y_0 \in Y$. Prove that

$$i_{y_0} : X \to X \times Y, \quad i_{y_0}(x) = (x, y_0)$$

is an embedding.

($i_{y_0}^{-1}|_{X \times \{y_0\}} = \pi_1 \circ i : X \times \{y_0\} \xrightarrow{i} X \times Y \xrightarrow{\pi_1} X$.
Moreover, π_1 is open map: if A open in $X \times Y$, take $x \in \pi_1(A)$, then $\exists\, y \in Y$ st $(x,y) \in A$, then $\exists\, U \ni x$ open, $\exists\, V \ni y$ open st $(x,y) \in U \times V \subseteq A$, then $x \in U \subseteq \pi_1(A)$. $\square$)

Ex. Let $f: X \times Y \to Z$ cont., fix $y_0 \in Y$.
Then $f_{y_0} := f \circ i_{y_0} : X \to Z$ is cont., but the converse is NOT true.
(Example: $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0) \\ \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0). \end{cases}$
Note that $f(1/n, 1/n) = 1/2$ for any $n \in \mathbb{N} \setminus \{0\}$.)

Ex Let $f: Z \to X \times Y$, $f(z) = (f_1(z), f_2(z))$ where $f_1 = \pi_1 \circ f : Z \to X$, $f_2 = \pi_2 \circ f : Z \to Y$.

Then: f cont $\iff$ f_1 and f_2 cont.

(($\Rightarrow$) Note that $f_i = \pi_i \circ f$; ($\Leftarrow$) For any subbasis element $S = \pi_i^{-1}(U) \in \mathcal{S}$, we have $f^{-1}(S) = f^{-1}(\pi_i^{-1}(U)) = f_i^{-1}(U)$) $\square$

EX. Let X be a top space with a countable basis. Then there exists D countable and dense (call X SEPARABLE).
[By AX. CHOICE, there exists D such that $D \cap B_n = \{d_n\}$ for $\{B_n\}_n$ basis.
Counterexample: take $\tau = \{A \subset X : x_0 \in A\} \cup \{\emptyset\}$, for $x_0 \in X$ fixed.
Then $\overline{\{x_0\}} = X$. Other example: $(\mathbb{R}, \tau_e)$.)

Ex. Let $f: Z \to \prod X_\alpha$, $f(z) = (f_\alpha(z))_\alpha$ where $f_\alpha = \pi_\alpha \circ f : Z \to X_\alpha$.
Then, for the product top:

$$f \text{ cont} \iff \forall \alpha,\ f_\alpha \text{ cont},$$

But it is NOT true for box top.

(Recall $f_\alpha = \pi_\alpha \circ f$ & $\forall s \in \mathcal{S}$, $f^{-1}(S) = f^{-1}(\pi_\beta^{-1}(U)) = f_\beta^{-1}(U)$.
For box top, the counterexample can be constructed as:

$$\mathbb{R}^{\mathbb{N}} = \{\text{sequences } (x_n)_n \subseteq \mathbb{R}\}, \quad f: \mathbb{R} \to \mathbb{R}^{\mathbb{N}},$$
$$f(t) = (t, t, \ldots)$$

Each $f_n(t) = t$ is cont, but f is not cont wrt box top:
take $B = (-1,1) \times (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{3}, \frac{1}{3}) \times \cdots \in \mathcal{B}_{BOX}$,
then $f^{-1}(B)$ is not open in $\mathbb{R}$: otherwise, $(-\delta, \delta) \subseteq f^{-1}(B)$,
then $f(-\delta, \delta) \subseteq B$ for some $\delta > 0$, but $(-\delta, \delta) = \pi_n(f(-\delta, \delta))$
$\subseteq \pi_n(B) = (-1/n, 1/n)$, impossible. $\square$)

LECTURE 3

§ 20 – METRIC TOPOLOGY

DEF. A *metric* on a set X is a function $d: X \times X \to \mathbb{R}$ such that

(D1) $d(x,y) \geq 0, \ \forall x, y$
  and $d(x,y) = 0 \iff x = y$

(D2) $d(x,y) = d(y,x)$

(D3) $d(x,y) + d(y,z) \geq d(x,z)$

We define the ε-ball centered at x as: $B_d(x,\varepsilon) := \{y \in X : d(x,y) < \varepsilon\}$.

DEF. Let (X,d) be a metric space.
$\mathcal{B} = \{B(x,\varepsilon) : x \in X, \ \varepsilon > 0\}$ is a basis for a topology.
The induced top. is called the *metric topology* associated to d.

proof: (B1) $x \in B(x,\varepsilon), \ \forall \varepsilon > 0$

(B2) $\forall B(x,\varepsilon), \ \forall y \in B(x,\varepsilon), \ \exists \delta > 0$ such that $B(y,\delta) \subseteq B(x,\varepsilon)$.
Indeed, it is enough to choose $\delta := \varepsilon - d(x,y)$. □

RMK ~~Recall that~~ From the general theory: a set U is open in the metric topology iff
$\forall y \in U, \ \exists \delta > 0$ such that $B(y,\delta) \subseteq U$.

EX. • In a set X, the *discrete metric*

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

induces the discrete topology.

- In $\mathbb{R}$, the Euclidean metric $d(x,y) = |x-y|$ induces τ_{EUCL}.
- In $\mathbb{R}^2$, the Euclidean metric $d((x,y),(x',y')) = \sqrt{(x-x')^2 + (y-y')^2}$ induces τ_{EUCL}. Also the square distance $\rho((x,y),(x',y')) = \max\{|x-x'|, |y-y'|\}$ works.
- The indiscrete top. on a set X with $\# X \geq 2$ is not induced by any metric! We'll see later...

~~DEF A top space (X,τ) is called METRIZABLE if there exists a metric on X st it induces the topology τ.~~

DEF. A top. space (X,τ) is called <u>metrizable</u> if $\exists$ a metric d on X that induces the topology τ.

EX. • $\mathbb{R}^2$ (or $\mathbb{R}^n$) is metrizable,

• (X, τ_{IND}), with $\# X \geq 2$, is <u>not</u> metriz.

• (X, τ_{DISCR}) is metriz.

RMKS • The URYSOHN METRIZATION THM characterizes metrizable spaces.

• Note that we are interested in <u>TOPOLOGICAL</u> properties. For example, we'll see that <u>boundedness</u> is not topological!

RMK. If d, d' are metrics on X, τ, τ' the associated topologies, then:

τ' finer than $\tau \iff \forall x \in X, \forall \varepsilon > 0, \exists \delta > 0$ st $B_{d'}(x,\delta) \subseteq B_d(x,\varepsilon)$

~~THM $\mathbb{R}^{\mathbb{N}}$ (with product topology)~~

THM $\mathbb{R}^n$ is metrizable (with product top).

proof Consider either

$$d_2((x_1,\dots,x_n),(y_1,\dots,y_n)) = \sqrt{(x_1-y_1)^2+\dots+(x_n-y_n)^2}$$

or $$d_1((x_1,\dots,x_n),(y_1,\dots,y_n)) = |x_1-y_1|+\dots+|x_n-y_n|$$

or $$d_\infty((x_1,\dots,x_n),(y_1,\dots,y_n)) = \max\{|x_1-y_1|,\dots,|x_n-y_n|\}.$$

They are all metrics, inducing the same topology since

$$d_\infty \leq d_2 \leq d_1 \leq n\, d_\infty .$$

$\square$

THM $\mathbb{R}^{\mathbb{N}}$ is metrizable with product topology, but it is <u>not</u> with box top.

proof • <u>PRODUCT TOP</u>: For any factor $\mathbb{R}$, consider the <u>uniform bound</u> of the Euclidean metric: EX

$$\bar{d}(x,y) = \min\{|x-y|, 1\},$$

which is a metric inducing the same Euclidean top.

Consider: $D\left((x_n)_n, (y_n)_n\right) := \sup \left\{ \frac{\bar{d}(x_n, y_n)}{n} : n \in \mathbb{N} \right\}$.

We're going to prove that D is a metric inducing the product top.

Note that the ~~...~~ sup is ≤ 1.

For the triangle ineq, note that, for any n,

$$\bar{d}(x_n, z_n) \leq \bar{d}(x_n, y_n) + \bar{d}(y_n, z_n)$$

therefore $D((x_n)_n, (z_n)_n) = \sup \bar{d}(x_n, z_n)/n \leq \sup \frac{\bar{d}(x_n, y_n)}{n} + \sup \frac{\bar{d}(y_n, z_n)}{n}$

$= D((x_n)_n, (y_n)_n) + D((y_n)_n, (z_n)_n)$

We prove it induces the product top:

($\subseteq$) let U open in the metric top of D:

fix $x \in U$, let $\varepsilon > 0$ st $x \in B_D(x, \varepsilon) \subseteq U$.

Let $N \gg 1$ st $1/N < \varepsilon$. Take $V := (x_1 - \varepsilon, x_1 + \varepsilon) \times \dots \times (x_N - \varepsilon, x_N + \varepsilon) \times \mathbb{R} \times \mathbb{R} \times \dots$

Then $x \in V \subseteq B_D(x, \varepsilon)$. Indeed, for any $y \in V$,

$$D(x, y) = \sup \left\{ \frac{\bar{d}(x_1, y_1)}{1}, \frac{\bar{d}(x_2, y_2)}{2}, \dots, \frac{\bar{d}(x_N, y_N)}{N}, \frac{\bar{d}(x_{N+1}, y_{N+1})}{N+1}, \dots \right\}$$

$$\leq \max \left\{ \bar{d}(x_1, y_1), \frac{\bar{d}(x_2, y_2)}{2}, \dots, \frac{\bar{d}(x_N, y_N)}{N}, \frac{1}{N} \right\} \leq \varepsilon$$

($\supseteq$) Let $U := \prod U_i$ be a basis element for the product top, namely $U_i \subseteq \mathbb{R}$ open, and $U_i = \mathbb{R}$ for all but a finite number of i, say $i \in \{\alpha_1, \dots, \alpha_n\}$.

For such finitely many i, let ε_i be st $x_i \in (x_i - \varepsilon_i, x_i + \varepsilon_i) \subseteq U_i$ and set $\varepsilon := \min\{ \varepsilon_i / i : i \in \{\alpha_1, \dots, \alpha_n\}\}$.

Then $x \in B_D(x, \varepsilon) \subseteq U$.

Finally, we show that $\mathbb{R}^{\mathbb{N}}$ is not metrizable with the box top.

LEMMA (SEQUENCE LEMMA).

> Let $A \subseteq X$ be a subset of a top. space.
> If $\exists (x_n)_n \subseteq A$ st $x_n \to x$, then $x \in \bar{A}$.
> When X is metrizable, also the converse holds true.

proof (Lemma) ($\Rightarrow$) $\forall U \ni x$ neighb, $\exists x_n \in U \cap A \neq \emptyset$ for $n \gg 1$, then $x \in \bar{A}$.

($\Leftarrow$) Let d be a metric inducing the top. For any n, since $B_d(x, 1/n) \cap A \neq \emptyset$, take $x_n \in B_d(x, 1/n) \cap A$. Since any open set $U \ni x$ contains an ε-ball, in particular $B_d(x, 1/n) \ni x_n$ for $n > 1/\varepsilon$. $\square$

We show that $(\mathbb{R}^{\mathbb{N}}, \tau_{BOX})$ does not satisfy the ~~...~~ conclusion of the sequence lemma.

Take $A = \{(x_n)_n : x_n > 0\}$. Any neighb of 0 intersect A, so $0 \in \overline{A}$.

But there is no sequence $(a_n)_n \subseteq A$ converging to 0.

If $a^{(n)} = (x_1^{(n)}, x_2^{(n)}, \ldots) \in A$, ie $x_i^{(n)} > 0$,

take $B' := (-x_1^{(1)}, x_1^{(1)}) \times (-x_2^{(2)}, x_2^{(2)}) \times \cdots$, then $0 \in B'$

but ~~...~~ $\forall n$, $a^{(n)} \notin B'$, so $(a^{(n)}) \not\to 0$. $\square$

THM $\mathbb{R}^{\mathbb{R}}$ is not metrizable, with product top.

proof We show ~~that~~ sequence lemma does not hold.

Choose $A = \{(x_\alpha)_\alpha$ st $x_\alpha = 1$ for all but finitely-many $\alpha\}$.

Note that $0 \in \overline{A}$: if $\prod U_\alpha \ni 0$, where the only proper open sets are $U_{\alpha_1}, \ldots, U_{\alpha_m}$, then take $x = (x_\alpha)_\alpha$ with $x_{\alpha_1} = \cdots = x_{\alpha_m} = 0$ and $x_i = 1$ for $i \notin \{\alpha_1, \ldots, \alpha_m\}$.

No sequence in A conv. to 0. Let $(a^{(n)})_n \in A$ and denote by $J_n := \{\alpha : a_\alpha^{(n)} \neq 1\}$, which is finite. Then $J = \bigcup_{n \in \mathbb{N}} J_n$ is countable. But $\mathbb{R}$ is not countable, then $\exists \beta \in \mathbb{R} \setminus J$.

~~...~~ It means that, $\forall n$, $a_\beta^{(n)} = 1$.

Consider $U := \pi_\beta^{-1}((-1,1)) \ni 0$, but $U \not\ni a_n$ for any n. $\square$

DEF. In $\mathbb{R}^{\mathbb{R}}$, define the metric
$$\delta(x,y) := \sup \{ \bar{d}(x_\alpha, y_\alpha) : \alpha \in \mathbb{R} \}$$
where $\bar{d} = \min\{d_{EUCL}, 1\}$.
It is a metric, called uniform metric. The induced topology is called uniform topology.

RMK More in general, it works for Y^X where X any set, (Y,d) metric space.

THM The uniform top on $\mathbb{R}^{\mathbb{R}}$ is finer than prod ~~and box~~ top and coarser than box top:
$$\tau_{BOX} \supseteq \tau_{UNIF} \supseteq \tau_{PROD},$$
and they are different.

proof • Let $\prod U_\alpha \ni x$ be a basis element for prod top, ie $U_\alpha \subseteq \mathbb{R}$ open, with $U_\alpha = \mathbb{R}$ except for finitely-many α.
For these α, take $\varepsilon_\alpha > 0$ st $B_{\bar{d}}(x_\alpha, \varepsilon_\alpha) \subset U_\alpha$.
Set $\varepsilon := \min\{\varepsilon_\alpha\} > 0$. Then $B_\delta(x, \varepsilon) \subseteq \prod U_\alpha$.
• Let $B := B_\delta(x, \varepsilon)$. Then $U := \prod_{\alpha \in \mathbb{R}} (x_\alpha - \frac{1}{2}\varepsilon, x_\alpha + \frac{1}{2}\varepsilon) \subseteq B$.
• On the other side:
$\prod B(x_\alpha, \varepsilon_\alpha)$ with $\varepsilon_\alpha \to 0$ is open in τ_{BOX}, not in τ_{UNIF};
$\prod B(x_\alpha, 1/2)$ is open in τ_{UNIF}, not in τ_{PROD}. $\square$

§ 21 – METRIC TOPOLOGY – CONTINUED

The following result characterizes continuity in metric spaces:

THM Let $f: X \to Y$, where X is metrizable, with metric d_X,
Y " " " d_Y.

Then: f cont $\iff$ $\forall x \in X, \forall \varepsilon > 0, \exists \delta > 0$ st
$$\left[d_X(x,y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon \right]$$

proof. ($\Rightarrow$) $f^{-1}(B_Y(f(x), \varepsilon)) \supseteq B(x, \delta)$ for some δ since it is open

($\Leftarrow$) for any $V \subset Y$ open, take $x \in f^{-1}(V)$. Since $f(x) \in V$, then $\exists\, \varepsilon > 0$ st $B(f(x), \varepsilon) \subseteq V$. By hypothesis, $\exists\, \delta > 0$ st $\forall y \in B(x, \delta)$, $f(y) \in B(f(x), \varepsilon)$, which is exactly saying that $x \in B(x, \delta) \subseteq f^{-1}(V)$.

□

We already met the:

LEMMA (SEQUENCE LEMMA).

Let $A \subseteq X$ be a subset of a top space X.

If $\exists\, (x_n)_n \subseteq A$ st $x_n \to x$, then $x \in \bar{A}$.

When X is metrizable, then also the converse holds true.

We can then characterize continuity in terms of convergent sequences:

THM Let $f: X \to Y$. If f is continuous, then, for any convergent sequence $x_n \to x$, then $f(x_n) \to f(x)$.

When X is metrizable, the converse holds true too.

proof ($\Rightarrow$) Let $V \ni f(x)$ open. Then $f^{-1}(V) \ni x$ neighb., so $\exists N$ st $x_n \in f^{-1}(V)$ for any $n \geq N$, then $f(x_n) \in V$ for any $n \geq N$.

($\Leftarrow$) We show that $f(\bar{A}) \subseteq \overline{f(A)}$; it is enough to have f cont.
If $x \in \bar{A}$, then $\exists\, (x_n)_n \to x$ by LEMMA. Then $f(x_n) \to f(x)$. Then $f(x) \in \overline{f(A)}$.

□

RMK We only used that X has a countable basis at the point x, meaning that there exists $\{U_n\}_{n\in\mathbb{N}}$ countable collection of neighbourhoods of x st, $\forall U \ni x$ neigh, $\exists n$ such that $x \in U_n \subseteq U$.
In this case, we say that X has the FIRST COUNTABILITY AXIOM.

DEF. Let $(f_n : X \to Y)_n$ be a sequence of functions with values in a metric space (Y, d). Recall that Y^X is endowed with the uniform metric, whose induced top. is called the uniform topology.
When $(f_n)_n$ converges to $f : X \to Y$ in the uniform top, we say that $(f_n)_n$ converges uniformly to f.
It can be characterized as:

$$\forall \varepsilon > 0, \exists N \text{ st } \forall n > N, \forall x \in X, d(f_n(x), f(x)) < \varepsilon.$$

THM. If $(f_n : X \to Y)_n$ are continuous functions, X top. sp., with values in (Y,d) and $(f_n)_n$ converges to f uniformly, then f is also cont.

proof. Let $V \subset Y$ open and $x_0 \in f^{-1}(V)$, $y_0 = f(x_0)$.
Let $\varepsilon > 0$ st $B(y_0, \varepsilon/3) \subset V$. By uniform convergence, $\exists N \gg 1$ st $\forall n \geq N$, $\forall x \in X$, $d(f_n(x), f(x)) < \varepsilon/3$.
Since f_N cont, $\exists U \ni x_0$ st $f_N(U) \subseteq B(f_N(x_0), \varepsilon/3)$.
Then: $f(U) \subseteq B(y_0, \varepsilon)$. Indeed,

$$d(f(x), y) \leq d(f(x), f_N(x)) + d(f_N(x), f_N(x_0)) + d(f_N(x_0), f(x_0)) < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$$

$\square$

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TUTORIAL 3

DEF. Let (X,d) be a metric space and $A \subseteq X$ subset.
Say A is BOUNDED if $\exists M>0$ st $\forall a,b \in A$, $d(a,b) \leq M$.
If A is bounded and non-empty, its DIAMETER is
$$\operatorname{diam} A := \sup\{d(a,b) : a,b \in A\}$$

EX Let (X,d) be a metric space. Define the STANDARD BOUNDED METRIC
$$\bar{d}(x,y) := \min\{d(x,y), 1\}.$$
Then $\bar{d}$ is a metric inducing the same top. as d.
In particular, being bounded is not a top. property.

(For the triangle inequality, notes that $\bar{d}(x,z) \leq \bar{d}(x,y) + \bar{d}(y,z)$ whenever $d(x,y) \geq 1$ or $d(y,z) \geq 1$. Otherwise, $\bar{d}(x,z) \leq d(x,z) \leq d(x,y) + d(y,z) = \bar{d}(x,y) + \bar{d}(y,z)$, using triangle ineq. for d.
For the second statement, note that $\{B(x,\varepsilon) : \varepsilon < 1\}$ is a basis in any metric space.)

EX $d : X \times X \to \mathbb{R}$ is continuous

(Fix $(x_0,y_0) \in X \times X$. For any $x \in B(x_0,r)$, for any $y \in B(y_0,r)$, then $d(x,y) \leq d(x,x_0) + d(x_0,y_0) + d(y_0,y)$, then
$|d(x,y) - d(x_0,y_0)| \leq 2r$, meaning $d(x,y) \in \underset{\text{EUCL}}{B}(d(x_0,y_0), 2r)$.)

EX. In $\mathbb{Z}$, fix p prime. For any $n \in \mathbb{N}$, consider $n = p^{v(n)} \cdot m$, where $m \nmid p$. Define the p-ADIC DISTANCE
$$|0|_p = 0, \qquad |n|_p = p^{-v(n)},$$
$$\rightsquigarrow \quad d(n,m) = |n-m|_p.$$
(Let $d(n,m) = p^{-h}$, ie $n-m = p^{h} \cdot s$; $d(m,q) = p^{k}$, ie $m-q = p^{k} t$.
Set $j := \min\{h,k\}$, so $p^{j} \mid (n-m)+(m-q) = n-q$. Then $v(n-q) \geq j$ so $d(n-q) = p^{-v(n-q)} \leq p^{-\min\{h,k\}} = \max\{p^{-h}, p^{-k}\}$
$\leq p^{-h} + p^{-k} = d(n,m) + d(m,q)$.)

Ex Let (X,d) be a metric space.
Then it is HAUSDORFF and T1 PROPERTY.
In particular, $\mathbb{R}_{usc}$ is ~~not~~ metrizable.

Ex If X is metrizable and separable, then it has a countable basis.

(We show that $\{B(d,q) : d \in D, q \in \mathbb{Q}^{>0}\}$, for D countable dense, is a basis. Indeed, if A open, $\forall x \in A$, $\exists r>0$ st $B(x,r) \subseteq A$. Let $q \in \mathbb{Q}$ be st $q < r/2$. Since D dense, $\exists d \in D \cap B(x,q)$. Then, $\forall z \in B(d,q)$, $d(x,z) \leq q+q < r$, so $z \in B(x,r)$.)

Ex $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is not metrizable.
(Note that, if X is metrizable, any subset still is. Note also that $\mathbb{Q}$ is dense in $\mathbb{R}_\ell$, therefore $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is separable. If it was metrizable, then it would have a countable basis. Then any subspace would have a countable basis too. Consider $Y = \{(x,y) \in \mathbb{R}^2 : x+y=1\}$. It has the discrete top, since $([x, x+1) \times [1-x, 2-x)) \cap Y = \{x, 1-x\}$.)

Ex. Let (X,d_X), (Y,d_Y) be metric spaces. A function $f: X \to Y$ is called LIPSCHITZ if $\exists L>0$ st:
$$\forall x,y \in X, \quad d_Y(f(x), f(y)) \leq L \cdot d_X(x,y).$$
Then: Lipschitz $\Rightarrow$ continuous.
In particular, isometric spaces are homeomorphic.
Recall: $f:(X,d_X) \to (Y,d_Y)$ is a (LOCAL) ISOMETRY if, $\forall x,y \in X$, $d_Y(f(x), f(y)) = d_X(x,y)$. Two metric spaces are ISOMETRIC if there exists a bijective isometry.

(Note that $f(B_{d_X}(x, r/L)) \subseteq B_{d_Y}(f(x), r)$.
Also prove isometries are continuous and injectives.)

<u>EX</u> Determine the isometries of $\mathbb{R}$.

(<u>CLAIM</u>: $f(x) = \pm x + a$, for $a \in \mathbb{R}$.

Set $f(0) =: a$. The translation $T_a : x \mapsto x - a$ is an isometry, so $f' := T_{-a} \circ f$ is an isom. st $f'(0) = 0$.

Since $|f'(1)| = d(f'(0), f'(1)) = d(0,1) = 1$, we get $f'(1) = \{-1, 1\}$.

In case $f'(1) = -1$, then consider $r : x \mapsto -x$ isometry and set $f'' := r \circ f'$ so that $f''(1) = 1$. Otherwise just set $f'' = f'$.

Then $|f''(x)| = d(f''(x), f''(0)) = d(x,0) = |x|$. Then $(f''(x))^2 = x^2$, Moreover $|f''(x) - 1| = |x - 1|$, then $f''(x)^2 - 2f''(x) + 1 = x^2 - 2x + 1$, whence $f''(x) = x$. $\square$)

<u>EX</u> Let (X, d) be a metric space, $Z \subseteq X$, $Z \neq \emptyset$. The function

$$d_Z : X \to \mathbb{R}, \quad d_Z(x) = \inf_{z \in Z} d(x, z)$$

is continuous. In particular, $F := \{x : d_Z(x) = 0\}$ is closed.

($\forall \varepsilon > 0$, $\exists z$ st $d(x, z) \leq d_Z(x) + \varepsilon$. Then $d_Z(y) \leq d(y, z) \leq d(y, x) + d(x, z)$ $\leq d(y, x) + d_Z(x) + \varepsilon$. Then $d_Z(y) - d_Z(x) \leq d(y, x) + \varepsilon$ for any ε.

Then $|d_Z(y) - d_Z(x)| \leq d(x, y)$.)

<u>EX.</u> On a metric space (X, d), define

$$d'(x, y) := \frac{d(x, y)}{1 + d(x, y)}.$$

It is a metric, bounded, topologically equivalent to d, meaning they induce the same topology.

(Consider $f(t) = \frac{t}{1+t}$, so that $d'(x,y) = f(d(x,y))$.

Note that $f : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ and $f^{-1}(0) = \{0\}$ which ensures $d'(x,y) \geq 0$ and "$=0$" iff $x = y$. Moreover, f is bounded, so is d'.

Note $f' = \frac{1}{(1+t)^2}$, $f'' = -\frac{2}{(1+t)^3}$: so f is increasing and concave.

Concavity yields : $\forall \lambda$, $\lambda f(t) = (1-\lambda) f(0) + \lambda f(t) \leq f((1-\lambda) 0 + \lambda t) = f(\lambda t)$.

Since increasing : $\forall\, 0 \leq c \leq a + b$, then $f(c) \leq f(a+b) = \frac{a}{a+b} f(a+b) + \frac{b}{a+b} f(a+b) \leq f(a) + f(b)$

Then d' is a distance.

Lastly : $B_d(x, \varepsilon) \subseteq B_{d'}(x, \varepsilon)$ and $B_{d'}(x, \frac{\varepsilon}{\varepsilon+1}) \subseteq B_d(x, \varepsilon)$.)

Ex Is the Golomb topology on $\mathbb{Z}$ metrizable?
(Recall open neighb of a are $N_{a,b} = \{a + nb : n \in \mathbb{Z}\}$, for $b \in \mathbb{N} \setminus \{0\}$.
Note in a metric space $\overline{B(x,r)} \subseteq \{z \in X : d(x,z) \leq r\}$.
But $\overline{N_{a,b}} \cap \overline{N_{c,d}} \ni bd$, so any open set A, B has the property $\overline{A} \cap \overline{B} \neq \emptyset$.)

Ex. In (X,d) metric space:

(1) $\overline{B(x,r)} \overset{?}{=} D(x,r) := \{y : d(x,y) \leq r\}$?

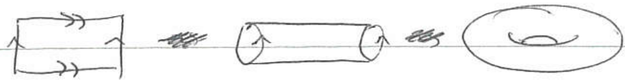
(2) Prove that, if $\overline{B(x,r)} = D(x,r)$, then there exists no non-trivial clopen.

(The inclusion $(\subseteq)$ is clearly true, but $(=)$ is not true in generale: considere d_{discr}, so ~~B(x,1)~~ $D(x,1) = X$ but $\overline{B(x,1)} = \overline{\{x\}} = \{x\}$.
For (2): let $A \neq \emptyset$ clopen, $A \neq X$. Then $B = A^c$ is also clopen, with $A \cap B = \emptyset$, $A \cup B = X$. Let $r = d(A,B) > 0$ (Ex!!). Let $a \in A$, $b \in B$, st $d(a,b) = d(A,B) = r$. Then $b \in D(a,r) = \overline{B(a,r)}$, then $B(b, r/2) \cap B(a,r) \neq \emptyset$. Say $\exists u \in B(b, r/2) \cap B(a,r)$; if $u \in A$, then $d(u,b) \geq r$; if $u \in B$ then $d(u,a) \geq r$; impossible!)

LECTURE 4

§ 22 — QUOTIENT TOPOLOGY

We give a mathematical tool to model "cut & paste" ~~tool~~ construction techniques.



DEF. A surj. map $p: X \to Y$ between top spaces is called a <u>QUOTIENT MAP</u> if:

$$U \text{ open in } Y \iff p^{-1}(U) \text{ is open in } X$$

RMK. • We say that a subset $C \subseteq X$ is <u>SATURATED</u> if:

$$\forall x \in C, \; p^{-1}(p(x)) \subseteq C.$$

Then p is quotient map $\iff$ $\begin{cases} \cdot\, p \text{ continuous} \\ \cdot \text{ and the image of } \underline{\text{saturated}} \text{ open sets is open} \end{cases}$

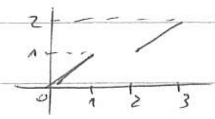
• A function $f: X \to Y$ is called <u>OPEN</u> if $\forall A \in \tau_X$, $f(A) \in \tau_Y$.
(or CLOSED if $\forall A$ closed in X, $f(A)$ closed in Y). Then:

$$f \text{ homeo} \Rightarrow \begin{Bmatrix} f \text{ open,} \\ \text{surj,} \\ \text{cont} \end{Bmatrix} \underset{*}{\Rightarrow} f \text{ quotient} \Rightarrow f \text{ cont.}$$

DEF. Let X be a top space, A a set, $p: X \to A$ surj.
There exists exactly one topology on A such that p is a quotient map. It is called the <u>QUOTIENT TOP</u> wrt p.

(It is enough to check that $\tau := \{U \subseteq A : p^{-1}(U) \text{ open in } X\}$ is a topology, which holds because $p^{-1}(\cap U_\alpha) = \cap p^{-1}(U_\alpha)$, $p^{-1}(\cup U_\alpha) = \cup p^{-1}(U_\alpha)$)

* RMK. The map $p: [0,1] \cup [2,3] \to p(x) \subseteq \mathbb{R}$, $p(x) = \begin{cases} x & \text{for } x \in [0,1] \\ x-1 & \text{for } x \in [2,3] \end{cases}$



is surj, closed, cont (so quotient), but <u>not</u> open.

• The map proj $\pi_1: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is cont, surj, open, but <u>not</u> closed:
eg $C = \{xy = 1\}$ is closed but $\pi_1(C) = \mathbb{R} - \{0\}$ is not.

DEF. Let X be a top space and X^* be a partition of X into disjoint subsets whose union is X. This corresponds to have an equivalence relation $\sim$ on X and taking $X^* = X/\sim$. Let $p: X \to X^*$ the surj map sending each point to the subset containing it. The space X^* with the quotient topology is called <u>quotient space</u>.

THM Let $p: X \to Y$ be a quotient map, let $A \subseteq X$ be a subspace that is saturated wrt p. Consider

$$q = p|_A : A \to p(A).$$

(i) If A is open, then q is a quotient map.
(ii) If p is open, then q is a quotient map.
(Same for closed).

proof [<u>A open</u>]. Take $V \subseteq p(A)$ st $q^{-1}(V)$ is open in A. We want to prove that V is open in $p(A)$. Since A is open in X and $q^{-1}(V)$ is open in A, then $q^{-1}(V)$ is open in X. But $q^{-1}(V) = p^{-1}(V)$ since A is saturated. Since p is quotient, then V is open in Y, then it is open in $p(A)$.

[<u>p open</u>] Take $V \subseteq p(A)$ st $q^{-1}(V)$ is open in A. Since A is saturated, then $q^{-1}(V) = p^{-1}(V) = U \cap A$ for some U open in X. Since p is surj, then $V = p(p^{-1}(V)) = p(U \cap A)$ $= p(U) \cap p(A)$. The last equality holds because: "$\subseteq$": always true; "$\supseteq$": since A saturated, if $y = p(a) = p(u)$ for some $a \in A$, $u \in U$, then $u \in p^{-1}(p(a)) \in A$. $\square$

RMK • We will see that T1 PROPERTY and HAUSDORFF PROPERTY are not preserved under quotient. (EX)

THM (UNIVERSAL PROPERTY).

Let $p: X \to Y$ be a quotient map.
Let $g: X \to Z$ be a map that is constant on each $p^{-1}(y)$.
Then g induces a map $f: Y \to Z$ such that $f \circ p = g$.
Moreover, f cont iff g cont; and f quotient iff g quot.

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "\forall g"] \arrow[dr, phantom, "\circlearrowleft" near start] & \\ Y \arrow[r, dashed, "\exists! f"'] & Z \end{tikzcd}$$

proof. • We define $f(y) := g(p^{-1}(\{y\}))$, since g is constant on $p^{-1}(y)$.

• If f cont $\Rightarrow$ $g = f \circ p$ still cont.
If g cont $\Rightarrow$ $\forall V \subseteq Z$ open, $g^{-1}(V) = p^{-1}(f^{-1}(V))$ is open;
since p quotient, then $f^{-1}(V)$ open.

• If f quot $\Rightarrow$ $g = f \circ p$ quotient.
If g quot $\Rightarrow$ g surj $\Rightarrow$ f surj. Moreover, for $V \subseteq Z$ st $f^{-1}(V)$ open in Y; since p cont, $g^{-1}(V) = p^{-1}(f^{-1}(V))$ is open in X. Since g quot, then V open in Z.

$\square$

COR. Let $g: X \to Z$ surj and cont.
Let $X^* = \{ g^{-1}(\{z\}) : z \in Z \}$ with quotient top.
Then: (a) g induces $f: X^* \to Z$ bijective and cont.
(b) g quotient $\Leftrightarrow$ $f: X^* \to Z$ homeom
(c) Z Hausdorff $\Rightarrow$ X^* Hausdorff

$$\begin{tikzcd} X \arrow[d] \arrow[dr, "g"] & \\ X^* \arrow[r, "f"'] & Z \end{tikzcd}$$

proof (a) By THM, it follows $\exists!\, f: X^* \to Z$ cont, bij
(b) ($\Rightarrow$) if g quotient, by THM also f quotient, then homeo
($\Leftarrow$) if f homeo, then f and p are quotient, then g quotient
(c) Take $x, y \in X^*$, $x \neq y$. Then $f(x) \neq f(y)$. Then $\exists U \ni f(x)$, $\exists V \ni f(y)$ disjoint open sets. Then $f^{-1}(U) \ni x$, $f^{-1}(V) \ni y$ are disjoint open sets.

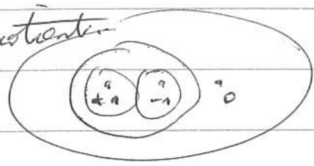
$\square$

TUTORIAL 4

Ex Let $p = \text{sign}: \mathbb{R} \to \{+1, 0, -1\}$, $p(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$.

ASS 2

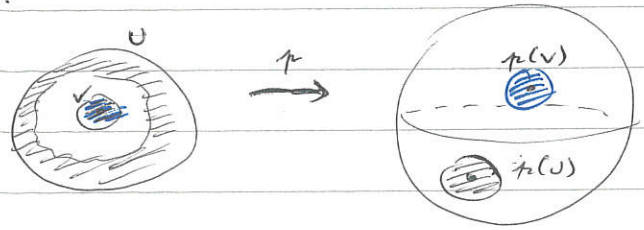
Determine the quotient top. on $\{-1, 0, +1\}$ wrt p.
In particular, T1 PROPERTY is not preserved by quotient.



(Open sets are: $\emptyset$, X, $\{1\}$, $\{-1\}$, $\{-1, 1\}$.
Note that there are not closed points, so T_1 fails)

Ex $X = \{(x,y) : x^2+y^2 \leq 1\} \subset \mathbb{R}^2$,
$X^* = \{\{(x,y)\} : x^2+y^2 < 1\} \cup \{S^1 = \{x^2+y^2=1\}\}$.
Describe the quotient top.

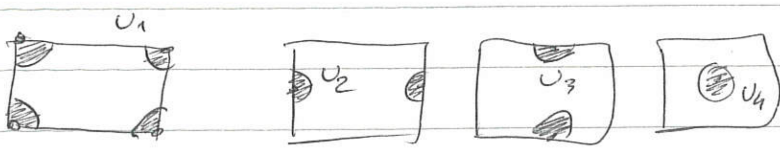
(Saturated open sets are:



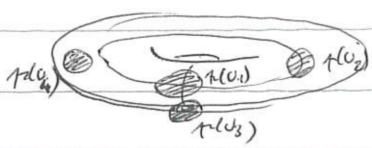
Ex $X = [0,1] \times [0,1] \subset \mathbb{R}^2$
$X^* = \{ \{(x,y)\} : 0 < x < 1, \ 0 < y < 1 \}$
$\cup \{ \{(x,0), (x,1)\} : 0 < x < 1 \} \cup \{ \{(0,y), (1,y)\} : 0 < y < 1 \}$
$\cup \{ \{(0,0), (0,1), (1,0), (1,1)\} \}$

Describe the quotient topology.

(Saturated open sets are:

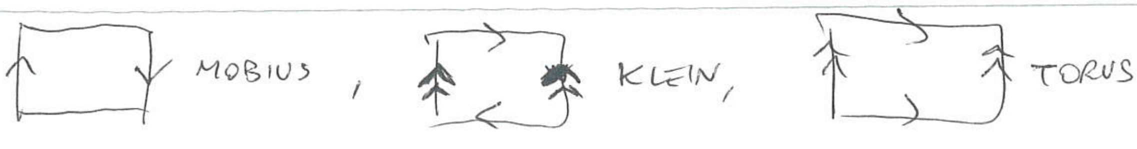


then we get



More precisely, consider:
$f: X \to S^1 \times S^1$, $f(s,t) = (e^{2\pi i s}, e^{2\pi i t})$
it yields a function $\tilde{f}: X^* \to S^1 \times S^1$ which is surj, inj, cont. and also homeom.)

Ex



<u>Ex</u> Consider $\mathbb{R}/\mathbb{Z}$, meaning $x \sim y \iff x-y \in \mathbb{Z}$.
Prove $\mathbb{R}/\mathbb{Z} \cong [0,1]/_{0\sim 1} \cong S^1$.

(Consider $f: [0,1] \xhookrightarrow{i} \mathbb{R} \xrightarrow{p} S^1$, $f(t) = \exp^{2\pi\sqrt{-1}t} = (\cos(2\pi t), \sin(2\pi t))$,
which is cont & surj. Since $f(t) = f(s) \iff s \sim t$, then
it induces a cont, bij map $\tilde{f}: [0,1]/_{0\sim 1} \longrightarrow S^1$.
We prove $\tilde{f}$ is also open (but f is not!).
For $A \subset [0,1]/_{0\sim 1}$ open, ie $A \subseteq (0,1)$ or $A \ni 0,1$ open,
then $p^{-1}(\tilde{f}(A)) = \bigoplus_{m \in \mathbb{N}} (A+m)$ is open in $\mathbb{R}$, then $\tilde{f}(A)$ is open in $\mathbb{R}/\mathbb{Z}$.
Therefore $\tilde{f}: [0,1]/_{0\sim 1} \longrightarrow \mathbb{R}/\mathbb{Z}$ is homeom.)

<u>RMK</u> We will see another argument: " $f: CPT \xrightarrow[cont]{} T_2$ is closed ".

<u>Ex</u> $\mathbb{R}/\mathbb{Q}$ (meaning $x \sim y \iff x-y \in \mathbb{Q}$) is indiscrete.
ASS 2
In particular, T_1 & HAUSDORFF are not preserved by quotient.
(If $A \neq \emptyset$ is open, then $p^{-1}(A) \subseteq \mathbb{R}$ open, then $\exists B(x,r) \subset p^{-1}(A)$.
Note that every equiv. class $[y]$ contains a point in $B(x,r)$:
Take $q \in \mathbb{Q}$ a good approxim of $x-y$, then $[y] = [y+q]$, with $y+q$ close to x. Then $\mathbb{R} \subseteq p^{-1}(A)$, so $A = \mathbb{R}/\mathbb{Q}$.)

<u>Ex.</u> If X Hausdorff and $\sim$ such that:

- $p: X \to X/\sim$ open,
- $R = \{(x,y) : x \sim y\}$ closed

then $X/\sim$ also Hausdorff

(If $[x] \neq [y]$, then $(x,y) \notin R$, then $\exists\, U \times V \ni (x,y)$ neigh st
$U \times V \subseteq R^c$, then $p(U), q(V)$ are open disjoint.)

EX On $\mathbb{R}$, consider $x \sim y \iff |x| = |y|$.
Prove that $\mathbb{R}/_\sim \cong [0, +\infty)$.

(Consider $g: \mathbb{R} \to [0,+\infty)$, $g(x) = |x|$ cont, surj.
Note $g(x) = g(y) \iff x \sim y$.
Then it induces $\tilde{g}: \mathbb{R}/_\sim \to [0,+\infty)$ cont., bij.
We prove $\tilde{g}$ is also open: if $A \subseteq \mathbb{R}/_\sim$ open, namely $p^{-1}(A) \subseteq \mathbb{R}$ open, saturated, then $p^{-1}(A)$ is union of intervals symmetric on the origin. Then $\tilde{g}(A) = g(p^{-1}(A)) = p^{-1}(A) \cap [0,+\infty)$ is an open subset of $[0,+\infty)$.)

EX. $\mathbb{R}/_\sim$ where $x \sim y \iff x = y$ or $x, y \in \mathbb{Z}$
is not FIRST COUNTABLE.

(Suppose $\exists$ $\{U_n\}_n \ni [0]$ countable local basis. For any n, take $[n - d_n, n + d_n] \subseteq p^{-1}(U_n)$. Let $U := (-\infty, 1/2) \cup \bigcup_n (n - d_n, n + d_n)$ $\subseteq \mathbb{R}$ neigh of $[0]$, which is not contained in any U_n.)

EX. $X = [-1,1]/_\sim$ where $x \sim y \iff |x| = |y| < 1$ or $x = y = 1$ or $x = y = -1$
Then X is T_1, but not Hausdorff.

EX $\mathbb{R}^n / GL(n;\mathbb{R})$ is not Hausdorff.

($GL(n;\mathbb{R})$ is a top. group, ie a group with a top st the group operation and the inverse map are continuous.
It acts on $\mathbb{R}^n$, ie $G \times \mathbb{R}^n \to \mathbb{R}^n$ cont st $g(h(x)) = (g \cdot h)(x)$, $id(x) = x$.
Note that $\mathbb{R}^n / GL(n;\mathbb{R}) \simeq \{\{0\}, \mathbb{R}^n - \{0\}\}$ where only $\{\mathbb{R}^n - \{0\}\}$ open.)

EX $\mathbb{R}^n / O(n) \simeq [0, +\infty)$
(Consider $f: \mathbb{R}^n \to [0,+\infty)$, $f(x) = \|x\|$.)

Ex. In $\mathbb{R}^{n+1} \setminus \{0\}$, consider: $x \sim y \Leftrightarrow \exists \lambda \in \mathbb{R} \setminus \{0\}$ st $y = \lambda x$.

Then $\mathbb{R}P^n := \mathbb{R}^{n+1} \setminus \{0\} / \sim$ is called PROJECTIVE SPACE.

~~The equivalence~~ The equivalence class $[x]$ represents the line through $0, x$, without the origin.

Note that $A_i := \{ [x_0 : \cdots : x_n] : x_i \neq 0 \} \simeq \mathbb{R}^n$ gives an open covering $\{A_i\}_i$ of $\mathbb{R}P^n$.

(Indeed, $p^{-1}(A_i)$ is a saturated open set.
Let $f : \mathbb{R}P^n \to A_0$, $[y_1 : \cdots : y_n] \mapsto (1, y_1, \dots, y_n)$, then
$f^{-1} \circ p : p^{-1}(A_0) \to \mathbb{R}^n$, $(x_0, \dots, x_n) \mapsto \left(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \dots, \frac{x_n}{x_n} \right)$ is cont,
then f^{-1} is cont. Then f is homeom.)

Ex. • $D^2 / \sim$, where $x \sim \pm x$ for $x \in S^1$, is homeom to $\mathbb{R}P^2$.

• $\mathbb{R}P^1 \simeq S^1$

(Use $g : S^1 \to S^1$ surj, open, cont, ~~def.~~ def. by $g(x,y) = g(z) = z^2 = (x^2 - y^2, 2xy)$.
Then it induces $\tilde{g} : S^1/\sim \to S^1$ homeom.)

LECTURE 5

§23 – CONNECTED SPACES

DEF. A top space X is CONNECTED if there is no pair of disjoint non-empty open sets U, V such that $X = U \cup V$.

RMK In other words: X connected $\Leftrightarrow$ the only clopen are $\emptyset$, X.

PROP. If $f: X \to Y$ cont, surj, X connected $\Rightarrow$ Y connected.

proof If $Y = U \cup V$, U, V open, $U, V \neq \emptyset$, $U \cap V = \emptyset$, then $X = f^{-1}(Y) = f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$, where $f^{-1}(U)$, $f^{-1}(V)$ are open and non-empty. Moreover, $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = \emptyset$. $\square$

COR "Being connected" is a topological property.

EX. $[0,1] \not\cong (0,1)$.

RMKS • $Y \subseteq X$ a subset is called CONNECTED if it is connected with the subspace top.

• $\mathbb{Q}$ is totally disconnected, meaning that any subset containing more than one point is disconnected.

PROP. Let X be a top space and $Y_i \subseteq X$ connected subsets.

- If $Y_i \cap Y_j \neq \emptyset$ $\forall i, j$, then $Y := \bigcup Y_i$ is connected.
- If $\exists Y_0$ st $Y_0 \cap Y_i \neq \emptyset$ $\forall i$, then $Y := \bigcup Y_i \cup Y_0$ is connected.

proof. CLAIM If $Y = U \cup V$ with U, V non-empty, open, disjoint, if $Z \subseteq Y$ connected, then either $Z \subseteq U$ or $Z \subseteq V$

(Indeed, $Z = Z \cap (U \cup V) = (Z \cap U) \cup (Z \cap V)$ with $Z \cap U$, $Z \cap V$ open, disjoint.)

- If $Y = U \cup V$ with U, V open, non-empty disjoint, then for any i, either $Y_i \subseteq U$ or $Y_i \subseteq V$. Since all share a point, they all are in U or V.
- Take $Z_i := Y_i \cup Y_0$ and use previous point. $\square$

EX $S^n = (S^n \setminus \{N\}) \cup (S^n \setminus \{S\})$ is connected,
where $S^n \setminus \{N\} \cong \mathbb{R}^n \cong S^n \setminus \{S\}$ via stereographic proj.

DEF Let X be top sp., $x_0 \in X$. The <u>CONNECTED COMPONENT</u> of x_0 is the largest connected subset containing x_0:

$$C(x_0) = \bigcup_{\substack{K \ni x_0 \\ K \text{ connected}}} K.$$

RMK
- X connected $\Leftrightarrow C(x_0) = X \quad \forall x_0$
- $\forall q \in \mathbb{Q}$, $C(q) = \{q\}$. In particular note that connected components are not necessarily open!

§ 24 – CONNECTED SUBSPACES OF $\mathbb{R}$

THM A subset of the real line is connected iff it is an interval.

proof • If Y is not an interval, then $\exists\, y_1, y_2 \in Y$, $\exists\, z \notin Y$ st $y_1 < z < y_2$. Then $Y = \big((-\infty, z) \cap Y\big) \cup \big((z, +\infty) \cap Y\big)$ disconnected.

• Suppose Y not connected, say $Y = A \cup B$ where A, B ___ . Take $x \in A$, $y \in B$, say $x < y$. Since Y interval, then $Y \supseteq [x, y]$.
Since $x \in A$, A open, then $\exists\, \delta > 0$ st $[x, x+\delta) \subseteq A$.
Let $T := \{ t \in [x, y] : [x, t) \subseteq A \}$. Since $x + \delta \in T$, then $T \neq \emptyset$. Define $\tau := \sup T \leq y$.
If $t \in A$, then $\exists\, \varepsilon > 0$ st $[t, t+\varepsilon) \subseteq A$, impossible.
If $t \in B$, then $\exists\, \varepsilon > 0$ st $(t - \varepsilon, t + \varepsilon) \subseteq B$, impossible.

RMK Up to homeom (use rational functions, exp, log, tan, atan) the connected subsets of $\mathbb{R}$ are:

$$\mathbb{R}, \qquad [0, +\infty), \qquad [0, 1].$$

$$\left(\begin{array}{l}
[a,b] \xrightarrow{\sim} [0,1] \quad \text{via } f(x) = \frac{x-a}{b-a} \\
(a,b) \underset{\sim}{\xrightarrow{f}} (0,1) \underset{\sim}{\xrightarrow{g}} \mathbb{R} \quad \text{via } g(y) = \tan(\pi y - \pi/2) \\
[a,b) \underset{\sim}{\xrightarrow{f}} [0,1) \underset{\sim}{\xrightarrow{h}} [0,+\infty) \quad \text{via } h(z) = e^{\frac{1}{1-z}} - e \\
(a,b] \underset{\sim}{\rightarrow} [a,b) \quad \text{via } x \mapsto -x \\
[a, +\infty) \xrightarrow{\sim} [0,1) \quad \text{via } y \mapsto 1 - e^{-(y-a)}
\end{array}\right)$$

COR. (INTERMEDIATE VALUE THEOREM)
Let $f : [a,b] \to \mathbb{R}$ continuous. Then $f([a,b])$ is an interval, ie, it takes any value between $\min f$, $\max f$.

§ 25 – COMPONENTS AND LOCAL-CONNECTEDNESS

DEF. $\alpha : [0,1] \to X$ cont is called PATH

X top space is PATH-CONNECTED if: $\forall x, y \in X$, $\exists \alpha$ path such that $\alpha(0) = x$, $\alpha(1) = y$.

EX
- $\mathbb{R}^n$: segment $\alpha(t) = (1-t)x + ty$
- S^n: if $x \neq \pm y$, take $\alpha(t) = \frac{(1-t)x + ty}{\|(1-t)x + ty\|}$ otherwise use intermediate point $z \neq \pm x$.
- Any CONVEX $S \subseteq \mathbb{R}^n$ (meaning: $\forall x, y \in S$, the segment $\overline{xy} \subset S$)
- Any STAR-SHAPED $S \subseteq \mathbb{R}^n$ (meaning: $\exists c$ center st $\forall x$, the segment $\overline{cx} \subseteq S$)
- for examples: ball

PROP. path-connected $\Rightarrow$ connected

proof Suppose $X = A \cup B$ disconnected. Take $x \in A$, $y \in B$, take α path connecting x and y. Then $\alpha^{-1}(A) \ni 0$, $\alpha^{-1}(B) \ni 1$ are open, non-empty, disjoint st $[0,1] = \alpha^{-1}(A) \cup \alpha^{-1}(B)$, impossible. $\square$

PROP. For $A \subseteq \mathbb{R}^n$ open subset of $\mathbb{R}^n$:

$$A \text{ connected} \Leftrightarrow A \text{ path connected}$$

proof Fix $p \in A$. The set $C := \{x \in A : \exists \alpha \text{ path joining } p \text{ and } x\}$ is non-empty, open, closed. $\square$

RMK In general, connected $\not\Rightarrow$ path-connected.

The counterex. is the TOPOLOGIST'S SINE:

$$X = \{(t, \sin(1/t)) : t > 0\} \cup \{0\}.$$

- The graph $\Gamma = \{(t, \sin(1/t))\}$ is connected because image of $\Phi = (id, f) : \mathbb{R}^{>0} \to \mathbb{R}$; X is connected because $\Gamma \subset X \subset \overline{\Gamma}$.
- X is not path connected: $\alpha(t) = (f(t), g(t)) \underset{(t>0)}{\in \Gamma}$, with $\alpha(0) = 0$ would give $\underset{\downarrow 0}{g(t)} = \sin\big(\underset{\downarrow +\infty}{1/f(t)}\big)$

RMK We can define the path components as equivalence classes of $x \sim y \Leftrightarrow \exists$ path joining x and y.
Path components are path-connected.

Ex. The topologist's sine has 1 conn component, but 2 path-comp.

DEF. X is called LOCALLY CONNECTED at x ~~if $\exists U \ni x$ connected~~ if: $\forall U \ni x$ neigh, $\exists V \ni x$ connected neighb s.t. $x \in V \subseteq U$.
(Similar def for locally path-connected.)

TUTORIAL 5

Ex. X connected $\iff$ $\forall A, B$ non-empty st $X = A \cup B$,
$$(A \cap \bar{B}) \cup (B \cap \bar{A}) \neq \emptyset$$

(($\Rightarrow$) If $\exists A, B \neq \emptyset$ st $X = A \cup B$, $(A \cap \bar{B}) \cup (B \cap \bar{A}) = \emptyset$, then
$\emptyset = (A \cup \bar{A}) \cap (A \cup B) \cap (\bar{A} \cup \bar{B}) \cap (B \cup \bar{B}) = \bar{A} \cap \bar{B}$ e $\bar{A} \cup \bar{B} = X$,
then $\bar{A}, \bar{B}$ are clopen, non-empty, disjoint, covering X.
($\Leftarrow$) If $X = A \cup B$ with A, B clopen, disjoint, non-empty, then
$(A \cap \bar{B}) \cup (B \cap \bar{A}) = A \cap B = \emptyset$.)

Ex. X is connected $\iff$ $\forall f : X \to \mathbb{Z}_2$ cont is constant.
Define $H(X; \mathbb{Z}_2) = \{ f : X \to \mathbb{Z}_2 \text{ cont} \}$ with group structure
$(f+g)(x) = f(x) + g(x) \bmod 2$. Then $H(X; \mathbb{Z}_2) = \mathbb{Z}_2^k$ where k is the number of connected components.

($X = A \cup B$ with A, B non-empty disjoint clopen $\leftrightarrow$ $f(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \in B \end{cases}$
non-const cont. function)

Ex X top space, $Y \subseteq X$ connected $\Rightarrow$ $\bar{Y}$ connected.

(More in general it holds $\forall Z$ st $Y \subseteq Z \subseteq \bar{Y}$. Let $Z = A \cup B$ disconnection.
Since Y is connected, then either $Y \subseteq A$ or $Y \subseteq B$. Say the former.
Since B open in Z, then $Z \setminus B$ closed in Z, then $Z \setminus B \supseteq \bar{Y}_{\cap Z}$. But
$Z \setminus B \subsetneq Z \subseteq \bar{Y}_{\cap Z}$, impossible.)

Ex A top space X is the disjoint union of its connected components.
Each connected component is closed (not necessarily open). The equivalence relation $x \sim y \iff C(x) = C(y)$ gives a quotient $X/\sim$ st:
it is T_1, tot disconnected. If ~~the~~ the number of connected components is finite, then $C(x)$ are clopen.

(Indeed, if $C(x)$ connected, then $\overline{C(x)}$ also connected.
Moreover, if $y \in C(x)$, then $C(y) \supseteq C(x)$ so $x \in C(y)$, then $C(x) = C(y)$, that is, equivalence classes are connected components.
If $C \subseteq X/\sim$ is connected, then $\pi^{-1}(C)$ is connected.)
~~[illegible]~~

<u>EX</u>. If X connected, then $X/\sim$ connected too.
If $X/\sim$ connected & equivalence classes connected, then X connected.
(First statement follows by continue image of ~~connected~~ connected is connected.
If $X = A \cup B$ disconnection, then each equivalence class is either in A or in B. ~~[illegible]~~ Then A, B are saturated (ie $A = \pi^{-1}(\pi(U)), \dots$), then $\pi(A), \pi(B)$ open disconnecting $X/\sim$.)

<u>EX</u> $\mathbb{R}P^n$, S^n are connected

($\mathbb{R}^{n+1} \setminus \{0\}$ is connected: take $A = \{x_n > 0\} \simeq \mathbb{R}^n$, $B = \{x_n < 0\} \simeq \mathbb{R}^n$, both connected. Then $\overline{A}, \overline{B}$ in $\mathbb{R}^n - \{0\}$ are connected in $\mathbb{R}^n - \{0\}$.
Moreover $\overline{A} \cap \overline{B} \neq \phi$, so $\overline{A} \cup \overline{B} = \mathbb{R}^n - \{0\}$ is connected.
Then the quotients $S^n = \mathbb{R}^{n+1} \setminus \{0\} / \mathbb{R}^{>0}$, $\mathbb{R}P^n = \mathbb{R}^{n+1} \setminus \{0\} / \mathbb{R}^*$ are connected.)

Ex $\prod_i X_i$ ~~conn~~ with prod top is conn $\Longleftrightarrow$ $\forall i$, X_i connected.

(Case I finite (eg $|I|=2$). • We claim that: $f: X \to Y$ cont, surj, open, Y connected, $f^{-1}(y)$ conn $\forall y$ implies X conn.

Indeed, if $X = A \cup B$ disconnection, then $Y = f(A) \cup f(B)$ where $f(A), f(B)$ open. Since Y connected, then $f(A) \cap f(B) \neq \phi$. Let $y \in f(A) \cap f(B)$. Then $f^{-1}(y) \cap A \neq \phi$, $f^{-1}(y) \cap B \neq \phi$, impossible.

• Recall the canonical proj^n are open. Indeed, for A open in $X \times Y$, take $x \in p(A)$, ie $(x,y) \in A$. Then $\exists B_1 \times B_2 \ni (x,y)$ open, $B_1 \times B_2 \subseteq A$, then $x \in B_1 \subseteq p(A)$.

Case I infinite. • Fix $p = (p_i)_i \in X$. Consider $Y = \{ (x_i)_i : x_i = p_i$ except for finitely-many $i \}$

• We claim Y connected. Indeed, $\forall F \subseteq I$ finite, set $Y_F = \{ (x_i)_i : x_i = p_i \ \forall i \notin F \} \simeq \prod_{i \in F} X_i$ is connected. Moreover, $Y = \bigcup_{\substack{F \subseteq I \\ \text{finite}}} Y_F$, where $p \in \bigcap_F Y_F \neq \phi$.

• We claim that Y dense. Indeed, for $A = \prod U_i$ open non-empty, meaning $U_i \subseteq X_i$ open non-empty, $U_i = X_i$ for $i \notin F$, F finite, let $q_i \in U_i$ for $i \in F$ and $q_i = p_i$ for $i \notin F$. Then $(q_i)_i \in A \cap Y_F \subseteq A \cap Y \neq \phi$. $\square$)

Ex $\mathbb{R}^{\mathbb{N}}$ with box top is not connected

($\mathbb{R}^{\mathbb{N}} = \{\text{bounded seq}\} \cup \{\text{unbounded sequences}\}$ are disjoint, non-empty open subsets: indeed, if $(a_i)_i$ is $\left\{ \begin{smallmatrix} \text{bounded} \\ \text{unbounded} \end{smallmatrix} \right\}$, then also any sequence in $\prod_i (a_i - 1, a_i + 1)$ is $\left\{ \begin{smallmatrix} \text{bounded} \\ \text{unbounded} \end{smallmatrix} \right\}$.)

<u>EX</u> $GL^+(n;\mathbb{R})$ is connected and path-connected

(• $GL^+(1;\mathbb{R}) = (0,+\infty)$.
For $n\geq 2$: consider $p: M(n\times n)\to \mathbb{R}^n$, $A\mapsto$ first column A^1.
It is open and continuous, since a projection. Since
GL^+ is open in M, then $p|_{GL^+}: GL^+\to\mathbb{R}^n$ still open (and cont.)
Since $\mathbb{R}^n$ conn. and fibres are omeom to $p^{-1}(1,0,\dots,0) \simeq GL^+(n-1)$
we conclude. (note $p(AB)=Ap(B)$)

• For path-connectedness. Let $A=UP$ polar decomposition, where U orthogonal, P symmetric, $P\geq 0$. Since SO is path-connected (see below), then $\exists\ u:[0,1]\to SO$ st $u(0)=U$, $u(1)=I$. By SPECTRAL THM, $P=M^t\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}M$. Take $\ell(t)=\begin{pmatrix}(1-t)\lambda_1+t & & 0\\ & \ddots & \\ 0 & & (1-t)\lambda_n+t\end{pmatrix}$
and set $\alpha(t):=u(t)M^t\ell(t)M$. $\square$)

<u>EX</u> SO path-connected

(Take canonical form $A=M^tBM$ where
$$B=\begin{pmatrix} I_k & & \\ & -I_h & \\ & & R_\ell\end{pmatrix},\quad R_i=\begin{pmatrix}\cos\alpha_i & -\sin\alpha_i\\ \sin\alpha_i & \cos\alpha_i\end{pmatrix}$$
Set $Q_{t}=\begin{pmatrix}\cos(1-t)\pi & -\sin(1-t)\pi\\ \sin(1-t)\pi & \cos(1-t)\pi\end{pmatrix}$, $R_{t,i}=\begin{pmatrix}\cos((1-t)\alpha_i) & -\sin((1-t)\alpha_i)\\ \sin((1-t)\alpha_i) & \cos((1-t)\alpha_i)\end{pmatrix}$

and set $A(t)=M^t\begin{pmatrix} I & & \\ & Q_{h,t} & \\ & & R_{i,t}\end{pmatrix}M^t$.

Ex $f: S^n \to \mathbb{R}$ cont. Then $\exists\, x \in S^n$ tc $f(x) = f(-x)$.

In particular, $\nexists$ open in $\mathbb{R}$ omeom to an open set of $\mathbb{R}^n$.

(Take $g(x) = f(x) - f(-x)$. Note that $g(S^n)$ is connected and $g(x) = -g(-x)$. Use intermediate value thm.
Note that any open set in $\mathbb{R}^n$ contain a ball then S^{n-1}.
Then the homeom φ would restrict to $\varphi|_{S^{n-1}} = S^{n-1} \to \mathbb{R}$ not injective.)

Ex. There exists no $f: \mathbb{R} \to \mathbb{R}$ cont st $f(\mathbb{Q}) \subseteq \mathbb{R} \setminus \mathbb{Q}$, $f(\mathbb{R} \setminus \mathbb{Q}) \subset \mathbb{Q}$.

~~(Note $f(\mathbb{Q})$ is countable also $f(\mathbb{R})$ is countable.~~
~~But $f($~~

(Note that $f(\mathbb{Q})$ is countable, $f(\mathbb{R} \setminus \mathbb{Q}) \subset \mathbb{Q}$ countable too, then $f(\mathbb{R})$ countable. Moreover, $f(\mathbb{R})$ connected, then interval. Then f constant.)

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